6

Plane Waves and Photons

6.1 Introduction

One of the most important solutions to the propagation equation in vacuum corresponds to plane waves, which are only functions of time and one spatial coordinate. In general, the phase of the plane is defined by the relativistically invariant quantity $\phi = k_\mu x^\mu = \mathbf{k} \cdot \mathbf{x} - \omega t$. Such plane waves represent classical solutions to the electromagnetic wave equation, but they assume a particularly important role in electrodynamics because they offer a natural introduction to the quantum electrodynamical concept of the photon. In addition, plane wave solutions form the basis of the Fourier analysis of wave propagation, which is a powerful mathematical tool to study linear problems where the principle of superposition applies. In this respect, plane waves are closely linked to the Green functions presented in Chapter 5 and represent their natural mathematical complement.

In this chapter, the vacuum propagation equation is first considered, and its plane wave solutions are described. We then consider the same type of waves but now confined in a box by a set of boundary conditions. The connection with the quantization of the free electromagnetic field and photons is then made, and a detailed analysis of the properties of the quantized electromagnetic field is given, including its statistical characteristics. Finally, virtual photons and the Coulomb field are discussed. The goal of this chapter is to provide a useful introduction to some aspects of QED and to offer a background for the question of vacuum fluctuations; this approach is also essential to the understanding of the Casimir effect and Hawking–Unruh radiation.

As discussed in Chapter 2, the propagation of electromagnetic waves in vacuum is described by the wave equation for the free electromagnetic field,

$$\partial_\mu \partial^\mu A_\nu = \Box A_\nu = 0, \tag{6.1}$$

where the four-potential satisfies the Lorentz gauge condition,

$$\partial_\mu A^\mu = 0, \tag{6.2}$$

and where $\partial_\mu = (-\partial_0, \nabla)$ is the four-gradient operator; $\Box = \partial_\mu \partial^\mu = \Delta - \partial_0^2$ is the d'Alembertian operator, also called the electromagnetic wave propagator.

We can seek a general solution of Equation 6.1 in terms of a superposition of plane waves, where the four-potential is described by a Fourier transform:

$$A_\mu(x^\lambda) = \left(\frac{1}{\sqrt{2\pi}}\right)^4 \iiiint \tilde{A}_\mu(k_\lambda) \exp(-ik_\lambda x^\lambda)\, d^4k_\lambda. \tag{6.3}$$

Here, $k_\mu = (\omega/c, \mathbf{k})$ is the four-wavenumber, which is the Fourier conjugate of the four-position, $x_\mu = (ct, \mathbf{x})$.

The terminology often used to describe the Fourier transform in Equation 6.3 refers to the four-wavenumber space as momentum space, since for quantum states, there is a direct correspondence between the four-momentum and four-wavenumber:

$$p_\mu = \hbar k_\mu. \tag{6.4}$$

The obvious advantage of the Fourier transform is that the d'Alembertian operator yields a very simple result when applied to the complex exponential in the Fourier integral: we now have, for the wave equation,

$$\Box A_\mu(x^\lambda) = \left(\frac{1}{\sqrt{2\pi}}\right)^4 \iiiint (-k_\mu k^\mu) \tilde{A}(k_\lambda) \exp(-ik_\lambda x^\lambda)\, d^4k_\lambda = 0. \tag{6.5}$$

At this point, it is important to note that the various Fourier modes are orthogonal; in other words,

$$\begin{aligned} \iiiint \exp(-ik_\lambda x^\lambda) \exp(ik'_\lambda x^\lambda)\, d^4x^\lambda &= \iiiint \exp[-i(k_\lambda - k'_\lambda)x^\lambda]\, d^4x^\lambda \\ &= (2\pi)^4 \delta^4(k_\lambda - k'_\lambda). \end{aligned} \tag{6.6}$$

As a result, the nontrivial solution to Equation 6.5 implies that the following condition be satisfied for all values of the four-wavenumber:

$$-k_\mu k^\mu = \frac{\omega^2}{c^2} - \mathbf{k}^2 = 0. \tag{6.7}$$

This is the well-known dispersion relation for electromagnetic waves propagating in vacuum. Within the context of the aforementioned momentum space terminology, we see that the vacuum dispersion relation corresponds to the mass-shell condition for photons, namely,

$$p_\mu p^\mu = \hbar^2 k_\mu k^\mu = 0 = \mathbf{p}^2 - \frac{E^2}{c^2}, \tag{6.8}$$

which implies that photons are massless, since all their energy corresponds to momentum, as opposed to a particle with rest mass m_0, for which $p_\mu p^\mu = -m_0^2c^2$. In the previous chapter, concerned with the propagation of electromagnetic waves in a vacuum enclosed by boundary conditions, we have seen that, in general, the presence of boundaries discretizes the free modes of the vacuum and introduces a cutoff frequency ω_0, such that the dispersion relation is modified to read $c^2k_\mu k^\mu + \omega_0^2 = 0$; within this context, the photons are seen to acquire an effective rest mass, $m_0 = \hbar\omega_0/c^2$. Indeed, the photons trapped at cutoff by the boundary conditions have energy but no momentum, which is exactly the definition of rest mass. However, a more detailed analysis of the situation shows that the trapped modes correspond to standing waves, which can be understood in terms of counterpropagating waves; thus, the zero momentum of the trapped wave stems from the fact that photon pairs with finite energy but opposite momenta form the cutoff mode.

Provided that the condition $k_\mu k^\mu = 0$ is satisfied, any superposition of plane waves described by Equation 6.5 is a solution of the wave equation in vacuum. A single free electromagnetic mode, described by the Fourier amplitude $\tilde{A}_\mu(k'_\lambda) = \tilde{A}_{\mu 0}\delta^4(k_\lambda - k'_\lambda)$, corresponds to a plane wave, with four-wavenumber k_λ, and the scalar $\phi = -k_\lambda x^\lambda$ is the relativistically invariant phase of that wave. In Chapter 8, where the dynamics of an electron in a plane wave is studied, we will generalize plane waves to include free electromagnetic modes where the four-potential is a function of the phase and satisfies the gauge condition, which now corresponds to the transversality condition. We have

$$A_\mu(x_\lambda) = A_\mu(\phi) = A_\mu(k^\lambda x_\lambda), \tag{6.9}$$

and the Lorentz gauge condition now requires that

$$\partial_\mu A^\mu(\phi) = \frac{\partial\phi}{\partial x^\mu}\frac{dA^\mu}{d\phi} = k_\mu\frac{dA^\mu}{d\phi} = 0. \tag{6.10}$$

In particular, if we choose the spatial z-axis to coincide with the direction of propagation of the wave, we see that a purely transverse four-potential, where $A_\mu(x_\lambda) = [0, \mathbf{A}_\perp(\phi), 0]$, will automatically satisfy the gauge condition.

6.2 Quantization of the Free Electromagnetic Field

We now turn our attention to the question of the quantization of the free electromagnetic field. So far, we have considered the electromagnetic field as a classical field, describable in terms of continuous functions of the four-position, or the four-wavenumber, when we work in momentum space. This model of the field is extremely useful when applied to a large number of phenomena, including wave propagation, diffraction, and interference. In particular, a large variety of optical phenomena can be described within this framework, which forms the basis of geometrical and classical optics, including coherent radiation generation in microwave devices, Fourier optics, and dispersion theory. However, under certain circumstances, this description proves inadequate, and the concept of the photon, or quantum of the electromagnetic field, must be introduced. Such special situations include the physics of vacuum fluctuations, the Casimir effect, and Hawking–Unruh radiation, which require a detailed knowledge of the statistical properties of the free electromagnetic field; the study of the coherence and space–time correlation characteristics of light, as exemplified by squeezed states; and the physics of radiative corrections which play a major role in Compton scattering, particularly in the case of multiphoton interactions at relativistic field intensities; finally, QED concepts, such as the Schwinger critical field and the description of the Coulomb field in terms of virtual quanta rely intrinsically on the concept of the photon.

We start from Maxwell's equations, as expressed in the absence of sources:

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}, \quad \text{and} \quad \nabla \cdot \mathbf{A} = 0, \tag{6.11}$$

for the source-free equations, and

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} = \mathbf{0}, \quad \text{and} \quad \nabla \cdot \mathbf{E} = 0, \tag{6.12}$$

in vacuum. In this case, we introduce the vector potential only, as we shall work in the Coulomb gauge. We simply have

$$\mathbf{E} = -\partial_t \mathbf{A}, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{6.13}$$

and the transversality condition, which is equivalent to the fact that the vector potential is divergence-free,

$$\nabla \cdot \mathbf{A} = 0. \tag{6.14}$$

We note that such a divergence-free vector potential will be used in Chapter 8, where the focusing and diffraction of electromagnetic waves will be studied in detail. In that case it will prove useful to introduce a generating vector field $\mathbf{G}$, defined such that $\mathbf{A} = \nabla \times \mathbf{G}$, thus automatically satisfying the Coulomb gauge condition described by Equation 6.14. Finally, using the definition of the fields in terms of the vector potential, we obtain the propagation equation,

$$\left(\Delta - \frac{1}{c^2}\partial_t^2\right)\mathbf{A} = \mathbf{0}. \tag{6.15}$$

To quantize the free electromagnetic field, we need to derive the corresponding Hamiltonian; it will prove useful to start by Fourier transforming the vector potential into three-momentum space, which is the conjugate of the usual three-dimensional space. In addition, we will consider that the field is subjected to spatial boundary conditions. Instead of a continuous spectrum, a discrete spectrum results from this procedure, and the vector potential can be described by a Fourier series. Finally, a cubic box is used, which further simplifies the mathematical expression of the potential. With this, we have

$$\mathbf{A}(x_\mu) = \frac{1}{\sqrt{\varepsilon_0 a^3}} \sum_\ell \sum_m \sum_n A_{\ell mn}(t) \exp\left[i(k_\ell x + k_m y + k_n z)\right], \tag{6.16}$$

where a is the size of the box, and where the wavenumber spectrum satisfies the periodic boundary conditions at the edge of the box:

$$k_\ell = \ell\frac{2\pi}{a}, \quad k_m = m\frac{2\pi}{a}, \quad k_n = n\frac{2\pi}{a}, \quad \ell, m, n \in \mathbb{Z}. \tag{6.17}$$

Finally, it is customary to introduce the normalization factor $1/\sqrt{\varepsilon_0 a^3}$ for convenience. Following Mandel and Wolf, in the remainder of the derivation, we will use a more compact notation, where Equation 6.16 now reads

$$\mathbf{A}(x_\mu) = \frac{1}{\sqrt{\varepsilon_0 a^3}} \sum \mathbf{A}_\mathbf{k}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.18}$$

and the summation is performed over the three spatial indices labeling the normal modes of the box; the wavenumber is defined as $\mathbf{k}_{\ell mn} = \mathbf{k} = \mathbf{k}_\ell \hat{x} + \mathbf{k}_m \hat{y} + \mathbf{k}_n \hat{z}$.

Within this context, the Coulomb gauge condition implies that

$$\frac{1}{\sqrt{\varepsilon_0 a^3}} \sum_\mathbf{k} i\mathbf{k} \cdot \tilde{\mathbf{A}}_\mathbf{k}(t) e^{i\mathbf{k}\cdot\mathbf{x}} = 0; \tag{6.19}$$

because the complex exponential functions are orthogonal, we must also have

$$\mathbf{k} \cdot \tilde{\mathbf{A}}_\mathbf{k}(t) = 0. \tag{6.20}$$

We can now use Equation 6.18 into the wave equation 6.15; the Laplacian operating on complex exponentials takes a very simple form, and we find that

$$\frac{1}{\sqrt{\varepsilon_0 a^3}} \sum_\mathbf{k} \left(-\mathbf{k}^2 - \frac{1}{c^2}\partial_t^2\right) \tilde{\mathbf{A}}_\mathbf{k}(t) e^{i\mathbf{k}\cdot\mathbf{x}} = 0. \tag{6.21}$$

Again, the orthogonality of complex exponentials implies that each term in the sum is identically equal to zero: each mode of the box satisfies the propagation equation

$$\left(\mathbf{k}^2 + \frac{1}{c^2}\partial_t^2\right)\tilde{\mathbf{A}}_{\mathbf{k}}(t) = 0. \tag{6.22}$$

Equation 6.22 corresponds to a harmonic oscillator of frequency $\frac{\omega^2}{c^2} = \mathbf{k}^2$ and can be solved to obtain

$$\tilde{\mathbf{A}}_{\mathbf{k}}(t) = a_{\mathbf{k}}e^{-i\omega t} + a^*_{-\mathbf{k}}e^{i\omega t}, \tag{6.23}$$

where the complex conjugate quantity guarantees that the vector potential is a real vector field.

The transversality condition described in Equation 6.20 must be satisfied by the solution given in Equation 6.23; it is customary to introduce two polarization vectors that are mutually orthogonal and perpendicular to the direction of propagation of the mode under consideration: $e_{\perp 1}^{\mathbf{k}}$ and $e_{\perp 2}^{\mathbf{k}}$ are two unit vectors defined such that $\mathbf{k} \cdot e_{\perp 1}^{\mathbf{k}} = \mathbf{k} \cdot e_{\perp 2}^{\mathbf{k}} = 0$; in addition, $e_{\perp 1}^{\mathbf{k}} \cdot e_{\perp 2}^{\mathbf{k}} = 0$; finally, $e_{\perp 1}^{\mathbf{k}} \times e_{\perp 2}^{\mathbf{k}} = \mathbf{k}/|\mathbf{k}| = e_{\parallel}^{\mathbf{k}}$. The vectors $(e_{\perp 1}^{\mathbf{k}}, e_{\perp 2}^{\mathbf{k}}, e_{\parallel}^{\mathbf{k}})$ form a right-handed, orthonormal basis, and the Fourier coefficients of the vector potential can be projected on this basis:

$$a_{\perp} = a_{\mathbf{k}1}e_{\perp 1}^{\mathbf{k}} + a_{\mathbf{k}2}e_{\perp 2}^{\mathbf{k}}. \tag{6.24}$$

We note that the polarization vectors can be rotated arbitrarily in the plane perpendicular to the direction of propagation of the mode under consideration; furthermore, the basis obviously depends on the mode, as it is defined with respect to $e_{\parallel}^{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. Within this context, the indices 1 and 2 represent the two possible polarization states of the mode indexed by $\mathbf{k}$; different combinations of the two vectors will correspond to linear, elliptical, and circular polarization states. For a detailed presentation of the polarization states and their properties under spatial rotations, we refer the reader to Mandel and Wolf.

The vector potential for an arbitrary combination of eigenmodes of the box can now be expressed as

$$\mathbf{A}(x^{\mu}) = \frac{1}{\sqrt{\varepsilon_0 a^3}}\sum_{\mathbf{k}}\sum_{\sigma}(a_{\mathbf{k}\sigma}e_{\perp\sigma}^{\mathbf{k}}e^{-i\omega t} + a_{-\mathbf{k}\sigma}e_{\perp\sigma}^{-\mathbf{k}}e^{i\omega t})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.25}$$

where $\sigma = 1, 2$ represents the polarization.

Although the presentation given here is not covariant, Equation 6.25 can be recast as

$$\mathbf{A}(x^{\mu}) = \frac{1}{\sqrt{\varepsilon_0 a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\left(a_{\mathbf{k}\sigma}e_{\perp\sigma}^{\mathbf{k}}e^{ik_{\mu}x^{\mu}} + a_{-\mathbf{k}\sigma}e_{\perp\sigma}^{-\mathbf{k}}e^{-ik_{\mu}x^{\mu}}\right), \tag{6.26}$$

where $k_\mu = (\omega/c, \mathbf{k})$, and satisfies the vacuum dispersion relation $k_\mu k^\mu = 0$ for each mode of the box.

Again, we emphasize that, strictly speaking, each mode of the box corresponds to an eigenwavenumber, $k_\mu^{\ell mn} = (\omega_{\ell mn}/c, \mathbf{k}_{\ell mn})$, satisfying the dispersion relation

$$\frac{\omega_{\ell mn}^2}{c^2} = \mathbf{k}_{\ell mn}^2 = (\ell^2 + m^2 + n^2)\left(\frac{2\pi}{a}\right)^2. \tag{6.27}$$

As mentioned earlier, for conciseness, the three indices corresponding to the triple set of boundary conditions are implicitly included in the summation $\sum_\mathbf{k}$. We also note that the oscillation frequency of a given eigenmode is independent of its polarization state; thus there is a degeneracy of the modes in terms of polarization.

To expand the vector potential in terms of the spatial eigenmodes of the box, we write $a_{\mathbf{k}\sigma}(t) = a_{\mathbf{k}\sigma}e^{-i\omega t}$, in which case Equation 6.25 takes the form

$$\mathbf{A}(x_\mu) = \frac{1}{\sqrt{\varepsilon_0 a^3}}\sum_\mathbf{k}\sum_\sigma [a_{\mathbf{k}\sigma}(t)e_{\perp\sigma}^\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}\sigma}^*(t)e_{\perp\sigma}^{-\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}]. \tag{6.28}$$

The fields can now be evaluated by using Equation 6.13; the partial time derivative and the curl operators take very simple forms, and we find

$$\mathbf{E}(x_\mu) = \frac{i}{\sqrt{\varepsilon_0 a^3}}\sum_\mathbf{k}\sum_\sigma \omega[a_{\mathbf{k}\sigma}(t)e_{\perp\sigma}^\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}} - a_{\mathbf{k}\sigma}^*(t)e_{\perp\sigma}^{-\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{6.29}$$

and

$$\mathbf{B}(x_\mu) = \frac{i}{\sqrt{\varepsilon_0 a^3}}\sum_\mathbf{k}\sum_\sigma \omega[a_{\mathbf{k}\sigma}(t)(\mathbf{k}\times e_{\perp\sigma}^\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a_{\mathbf{k}\sigma}^*(t)(\mathbf{k}\times e_{\perp\sigma}^{-\mathbf{k}})e^{-i\mathbf{k}\cdot\mathbf{x}}]. \tag{6.30}$$

To proceed with the quantization of the free electromagnetic field, we now need to derive the Hamiltonian of the system, which corresponds to the field energy of the eigenmodes derived above; we start with the electromagnetic energy density, which is given by $\frac{\varepsilon_0\mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2\mu_0}$. The total field energy in the box is thus

$$\mathcal{H} = \frac{1}{2}\iiint_{a^3}\left[\varepsilon_0\mathbf{E}^2(x_\mu) + \frac{1}{\mu_0}\mathbf{B}^2(x_\mu)\right]d^3\mathbf{x}, \tag{6.31}$$

where the electric field and magnetic induction are given by Equations 6.29 and 6.30. The orthogonality of the complex exponentials considerably simplifies the energy integral: the eigenmodes of the box do not interfere; in other words, we have

$$\iiint_{a^3} \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}]\,d^3\mathbf{x} = a^3\delta^3_{\mathbf{kk}'}. \tag{6.32}$$

For the magnetic field energy, we are led to evaluate the product

$$(\mathbf{k}\times e^{\mathbf{k}}_{\perp\sigma})\cdot(\mathbf{k}\times e^{-\mathbf{k}}_{\perp\sigma'}) = \mathbf{k}^2(e^{\mathbf{k}}_{\perp\sigma}\cdot e^{-\mathbf{k}}_{\perp\sigma'}) = \mathbf{k}^2\delta_{\sigma\sigma'}. \tag{6.33}$$

The aforementioned orthogonality of the box eigenmodes results in a diagonalization of the summations over modes and polarizations, and we find that the electromagnetic energy contained within the boundaries is

$$\mathcal{H} = 2\sum_{\mathbf{k}}\sum_{\sigma}\omega^2|a_{\mathbf{k}\sigma}(t)|^2; \tag{6.34}$$

as expected, the modes do not interfere. The total field energy is given by the sum of the energy of each of the vacuum eigenmodes excited within the box.

At this point, we need to describe the electromagnetic field within the context of Hamiltonian formalism; the commutation of the corresponding canonical variables will then enable us to quantize the field in terms of photons.

A generalized position, $q_{\mathbf{k}\sigma}$, and a generalized momentum, $p_{\mathbf{k}\sigma}$, must be associated to each eigenmode and polarization state of the system. Furthermore, these canonical variables must be such that the dynamics of the system obey Hamilton's equations,

$$\frac{\partial\mathcal{H}}{\partial p_{\mathbf{k}\sigma}} = \frac{\partial q_{\mathbf{k}\sigma}}{\partial t}, \tag{6.35}$$

and

$$\frac{\partial\mathcal{H}}{\partial q_{\mathbf{k}\sigma}} = \frac{\partial p_{\mathbf{k}\sigma}}{\partial t}. \tag{6.36}$$

The following choice of generalized coordinates yields the desired equations of motion:

$$q_{\mathbf{k}\sigma}(t) = [a_{\mathbf{k}\sigma}(t) + a^*_{\mathbf{k}\sigma}(t)], \tag{6.37}$$

$$p_{\mathbf{k}\sigma}(t) = -i\omega[a_{\mathbf{k}\sigma}(t) - a^*_{\mathbf{k}\sigma}(t)]. \tag{6.38}$$

Using the definition $a_{\mathbf{k}\sigma}(t) = a_{\mathbf{k}\sigma}e^{-i\omega t}$, it is easily seen that

$$\frac{\partial}{\partial t}[q_{\mathbf{k}\sigma}(t)] = p_{\mathbf{k}\sigma}(t). \tag{6.39}$$

In addition, taking the partial derivative of the generalized momentum with respect to time yields the following result:

$$\frac{\partial}{\partial t}[p_{\mathbf{k}\sigma}(t)] = -\omega^2 q_{\mathbf{k}\sigma}(t). \tag{6.40}$$

It is now easy to verify that the Hamiltonian can be expressed as

$$\mathcal{H}(p, q) = \frac{1}{2}\sum_{\mathbf{k}}\sum_{\sigma}[\omega^2 q_{\mathbf{k}\sigma}^2(t) + p_{\mathbf{k}\sigma}^2(t)], \tag{6.41}$$

and that Hamilton's equations 6.35 and 6.36 are satisfied by the choice of generalized coordinates used here. The fact that the eigenmodes are orthogonal is reflected in the fact that each mode contributes energy to the Hamiltonian independently of the other modes: there is no interference between modes. Furthermore, Equation 6.41 indicates that each mode corresponds to an harmonic oscillator, with a frequency satisfying the vacuum dispersion relation, Equation 6.26. In the quantization of the field, each oscillation mode will be identified with a quantum of radiation, thus introducing the concept of the photon.

Using Equations 6.36 and 6.38, we can define the amplitude of the box eigenmodes in terms of the canonical variables,

$$a_{\mathbf{k}\sigma}(t) = \frac{1}{2}\left[q_{\mathbf{k}\sigma}(t) + \frac{i}{\omega}p_{\mathbf{k}\sigma}(t)\right], \tag{6.42}$$

and the potential and fields can then be expressed in terms of the canonical variables:

$$\begin{aligned}\mathbf{A}(x_\mu) = \frac{1}{2\sqrt{\varepsilon_0 a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\bigg\{&\left[q_{\mathbf{k}\sigma}(t) + \frac{i}{\omega}p_{\mathbf{k}\sigma}(t)\right]e_{\perp\sigma}^{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} \\ &+ \left[q_{\mathbf{k}\sigma}^*(t) - \frac{i}{\omega}p_{\mathbf{k}\sigma}^*(t)\right]e_{\perp\sigma}^{-\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\bigg\},\end{aligned} \tag{6.43}$$

$$\begin{aligned}\mathbf{E}(x_\mu) = \frac{i}{2\sqrt{\varepsilon_0 a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\{&[\omega q_{\mathbf{k}\sigma}(t) + ip_{\mathbf{k}\sigma}(t)]e_{\perp\sigma}^{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} \\ &- [\omega q_{\mathbf{k}\sigma}^*(t) - ip_{\mathbf{k}\sigma}^*(t)]e_{\perp\sigma}^{-\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\},\end{aligned} \tag{6.44}$$

and

$$\mathbf{B}(x_\mu) = \frac{i}{2\sqrt{\varepsilon_0 a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\left\{\left[q_{\mathbf{k}\sigma}(t) + \frac{i}{\omega}p_{\mathbf{k}\sigma}(t)\right](\mathbf{k}\times e_{\perp\sigma}^{\mathbf{k}})e^{i\mathbf{k}\cdot\mathbf{x}} - \left[q_{\mathbf{k}\sigma}^{*}(t) - \frac{i}{\omega}p_{\mathbf{k}\sigma}^{*}(t)\right](\mathbf{k}\times e_{\perp\sigma}^{-\mathbf{k}})e^{-i\mathbf{k}\cdot\mathbf{x}}\right\}. \tag{6.45}$$

The quantum mechanical description of the electromagnetic field can be achieved by now considering the canonical variables as operators and identifying the corresponding Poisson bracket with the commutator:

$$[q_{\mathbf{k}\sigma}(t), p_{\mathbf{k}'\sigma'}(t)] = i\hbar\delta^3_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}. \tag{6.46}$$

Within the quantum context, the Hamiltonian of Equation 6.41 must now be considered as an operator acting on the state vector of the electromagnetic field; the result of this operation is the corresponding electromagnetic energy level. Because we are considering a system enclosed by a set of boundary conditions, the resulting energy spectrum is discrete. This is a very general result, and quantum numbers can be associated with each set of boundaries. In this case, each mode is indexed by three numbers corresponding to the triple set of spatial boundaries imposed on the electromagnetic field. For unbounded systems, the discrete spectrum is replaced by a continuum; for example, above the ionization threshold, the electron wavefunction can extend to infinity, and the energy spectrum becomes continuous. Additional quantum numbers can reflect internal symmetries, or, in the case of the electromagnetic field, indicate the state of polarization of the corresponding quantum state. The two different possible values for σ simply correspond to the fact that the photon is a spin-1 particle. The aforementioned degeneracy of the electromagnetic energy level in terms of the polarization illustrates the fact that the photon spin does not contribute to its energy.

6.3 Creation and Annihilation Operators

We now turn our attention to the well-known photon creation and annihilation operators; these can be defined in terms of the generalized position and momentum, $q_{\mathbf{k}\sigma}$, and $p_{\mathbf{k}\sigma}$, and are closely related to the eigenmode amplitudes, $a_{\mathbf{k}\sigma}$:

$$a_{\mathbf{k}\sigma}(t) = \frac{i}{\sqrt{2\hbar\omega}}[\omega q_{\mathbf{k}\sigma}(t) + ip_{\mathbf{k}\sigma}(t)] = \sqrt{\frac{2\omega}{\hbar}}\,a_{\mathbf{k}\sigma}(t), \tag{6.47}$$

and

$$a^{\dagger}_{\mathbf{k}\sigma}(t) = \frac{1}{\sqrt{2\hbar\omega}}[\omega q_{\mathbf{k}\sigma}(t) - ip_{\mathbf{k}\sigma}(t)] = \sqrt{\frac{2\omega}{\hbar}}\, a^{\dagger}_{\mathbf{k}\sigma}(t). \tag{6.48}$$

Here, the notation $a^{\dagger}_{\mathbf{k}\sigma}(t)$ refers to the Hermitian conjugate of the operator $a_{\mathbf{k}\sigma}(t)$; in terms of matrix properties, if $M = (M_{ij})$ is the original matrix, its Hermitian conjugate is given by $M^{\dagger} = (M^{*}_{ji})$. A matrix is said to be Hermitian if $M_{ij} = M^{*}_{ji}$; it is anti-Hermitian if $M_{ij} = -M^{*}_{ji}$; finally, the important property of unitarity is satisfied if $(M^{\dagger}M)_{ij} = \delta_{ij}$. Matrices and operators are closely related in quantum mechanics; in fact, the matrix element of a given operator corresponds to its projection in terms of quantum states: $M_{ij} = \langle i|M|j\rangle$, where we have used Dirac's notation.

Because the new operators introduced in Equations 6.46 and 6.48 are normalized, the commutation relation takes the simple form

$$[a_{\mathbf{k}\sigma}(t), a^{\dagger}_{\mathbf{k}'\sigma'}(t)] = \delta^{3}_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}; \tag{6.49}$$

in addition, it is easily verified that each operator commutes with itself.

Next, we need to express the Hamiltonian operator, which yields the energy spectrum of the quantized electromagnetic field in terms of the creation and annihilation operators. We start from Equation 6.41, and replace $q_{\mathbf{k}\sigma}(t)$ and $p_{\mathbf{k}\sigma}(t)$ by their expressions in terms of $a_{\mathbf{k}\sigma}(t)$ and $a^{\dagger}_{\mathbf{k}\sigma}(t)$. Adding Equations 6.46 and 6.48, we first have

$$\sqrt{\frac{2\omega}{\hbar}}\, q_{\mathbf{k}\sigma}(t) = [a_{\mathbf{k}\sigma}(t) + a^{\dagger}_{\mathbf{k}\sigma}(t)]. \tag{6.50}$$

Subtracting Equation 6.48 from Equation 6.46, we also find that

$$i\sqrt{\frac{2\omega}{\hbar}}\, p_{\mathbf{k}\sigma}(t) = [a_{\mathbf{k}\sigma}(t) - a^{\dagger}_{\mathbf{k}\sigma}(t)]. \tag{6.51}$$

Substituting in the definition of the Hamiltonian, given in Equation 6.41, we first have

$$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{\sigma}\left\{\frac{\hbar\omega}{2}[a_{\mathbf{k}\sigma}(t) + a^{\dagger}_{\mathbf{k}\sigma}(t)]^{2} - \frac{\hbar\omega}{2}[a_{\mathbf{k}\sigma}(t) - a^{\dagger}_{\mathbf{k}\sigma}(t)]^{2}\right\}. \tag{6.52}$$

Special care must now be taken in expanding the operator products in the brackets; in particular, their order must be preserved, and we can use the fact that each operator commutes with itself:

$$[a_{\mathbf{k}\sigma}(t) + a^{\dagger}_{\mathbf{k}\sigma}(t)]^{2} = 2a_{\mathbf{k}\sigma}(t)a^{\dagger}_{\mathbf{k}\sigma}(t), \tag{6.53}$$

and

$$[a_{\mathbf{k}\sigma}(t) - a^{\dagger}_{\mathbf{k}\sigma}(t)]^2 = -2a^{\dagger}_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t). \tag{6.54}$$

Substituting into the expression for the Hamiltonian, we find

$$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{\sigma}\hbar\omega[a_{\mathbf{k}\sigma}(t)a^{\dagger}_{\mathbf{k}\sigma}(t) + a^{\dagger}_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t)]. \tag{6.55}$$

Note that the operators are normalized so that the Hamiltonian has the correct dimension of energy, like the photon energy $\hbar\omega$.

To begin addressing the question of vacuum fluctuations, the Hamiltonian operator can be recast in a more suggestive form. We use the commutation rule given in Equation 6.49 and write it down explicitly:

$$[a_{\mathbf{k}\sigma}(t), a^{\dagger}_{\mathbf{k}'\sigma'}(t)] = \delta^3_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'} = a_{\mathbf{k}\sigma}(t)a^{\dagger}_{\mathbf{k}'\sigma'}(t) - a^{\dagger}_{\mathbf{k}\sigma}(t)a_{\mathbf{k}'\sigma'}(t). \tag{6.56}$$

We then have

$$a_{\mathbf{k}\sigma}(t)\,a^{\dagger}_{\mathbf{k}'\sigma'}(t) = a^{\dagger}_{\mathbf{k}\sigma}(t)a_{\mathbf{k}'\sigma'}(t) + \delta^3_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}, \tag{6.57}$$

and for $\mathbf{k}' = \mathbf{k}$, and $\sigma' = \sigma$, Equation 6.56 yields

$$a_{\mathbf{k}\sigma}(t)\,a^{\dagger}_{\mathbf{k}\sigma}(t) = a^{\dagger}_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t) + 1. \tag{6.58}$$

Finally, the Hamiltonian takes the form

$$\mathcal{H} = \sum_{\mathbf{k}}\sum_{\sigma}\hbar\omega\left[a^{\dagger}_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t) + \frac{1}{2}\right], \tag{6.59}$$

and the zero point fluctuations of the vacuum state appear clearly. The lowest energy level of each harmonic oscillator is $\frac{1}{2}\hbar\omega$, where the frequency is defined as the cutoff frequency of the mode in question: $\omega = \omega_{\ell mn} = \sqrt{\ell^2 + m^2 + n^2}(2\pi/a)$.

6.4 Energy and Number Spectra

The energy spectrum generated by the Hamiltonian operator can now be established. In this section, we will continue using Dirac's notation for quantum states and follow the discussion of Messiah, as presented by Mandel and Wolf. Before deriving the energy spectrum, we recall the definition of an operator eigenstate and the associated eigenvalue: for a given operator,

Ω, the quantum state $|\psi_n\rangle$ is an eigenstate with eigenvalue $\lambda_n \in \mathbb{C}$, if the following condition is satisfied

$$\Omega|\psi_n\rangle = \lambda_n|\psi_n\rangle. \tag{6.60}$$

In other words, the action of an operator on one of its eigenstates is very simple; the resulting state is just the original state multiplied by the eigenvalue. This concept is directly related to the diagonalization of the operator. If we consider orthonormalized eigenstates, the matrix elements of the operator take a diagonal form, with

$$\Omega_{mn} = \langle\psi_m|\Omega|\psi_n\rangle = \langle\psi_m|\lambda_n|\psi_n\rangle = \lambda_n\langle\psi_m|\psi_n\rangle = \lambda_n\delta_{mn}. \tag{6.61}$$

For an in-depth presentation of Dirac's notation, as well as Hilbert spaces and their application to quantum mechanics, we refer the reader to Messiah and to Cohen–Tannoudji, Diu, and Laloe.

In the Hamiltonian, the spectrum is governed by the operator $a^\dagger_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t) = N_{\mathbf{k}\sigma}(t)$; the factor $\frac{1}{2}\hbar\omega$ simply corresponds to the vacuum level and appears as a shift of the entire spectrum. Thus, our task is to determine the eigenvalue spectrum of the photon number operator, $N_{\mathbf{k}\sigma}(t)$. The mathematical approach we will follow here was originally presented by Dirac and starts by considering an eigenstate $|\eta_{\mathbf{k}\sigma}\rangle$ of the operator $N_{\mathbf{k}\sigma}(t)$, whereby

$$N_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle = n_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle, \tag{6.62}$$

and where $n_{\mathbf{k}\sigma}$ is the corresponding eigenvalue.

We now focus on the action of the creation operator on this eigenstate by evaluating the new quantum state $a^\dagger_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle$.

At this point, we need to derive the commutation relation between the photon number operator and the creation and annihilation operators:

$$\begin{aligned}[a_{\mathbf{k}\sigma}(t), N_{\mathbf{k}'\sigma'}(t)] &= a_{\mathbf{k}\sigma}(t)N_{\mathbf{k}'\sigma'}(t) - N_{\mathbf{k}'\sigma'}(t)a_{\mathbf{k}\sigma}(t)\\ &= a_{\mathbf{k}\sigma}(t)a^\dagger_{\mathbf{k}'\sigma'}(t)a_{\mathbf{k}'\sigma'}(t) - a^\dagger_{\mathbf{k}'\sigma'}(t)a_{\mathbf{k}'\sigma'}(t)a_{\mathbf{k}\sigma}(t),\end{aligned} \tag{6.63}$$

where we recognize the commutator described in Equation 6.56.

With this, we can rewrite Equation 6.63 as

$$[a_{\mathbf{k}\sigma}(t), N_{\mathbf{k}'\sigma'}(t)] = [a_{\mathbf{k}\sigma}(t), a^\dagger_{\mathbf{k}'\sigma'}(t)]a_{\mathbf{k}'\sigma'}(t) = \delta^3_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}a_{\mathbf{k}'\sigma'}(t). \tag{6.64}$$

We can proceed in the same way to evaluate

$$\begin{aligned}[a^\dagger_{\mathbf{k}\sigma}(t), N_{\mathbf{k}'\sigma'}(t)] &= a^\dagger_{\mathbf{k}\sigma}(t)N_{\mathbf{k}'\sigma'}(t) - N_{\mathbf{k}'\sigma'}(t)a^\dagger_{\mathbf{k}\sigma}(t)\\ &= a^\dagger_{\mathbf{k}\sigma}(t)a^\dagger_{\mathbf{k}'\sigma'}(t)a_{\mathbf{k}'\sigma'}(t) - a^\dagger_{\mathbf{k}'\sigma'}(t)a_{\mathbf{k}'\sigma'}(t)a^\dagger_{\mathbf{k}\sigma}(t).\end{aligned} \tag{6.65}$$

Again, we use the commutation relation between $a_{\mathbf{k}\sigma}(t)$ and $a^{\dagger}_{\mathbf{k}\sigma}(t)$ to obtain

$$[a^{\dagger}_{\mathbf{k}\sigma}(t), N_{\mathbf{k}'\sigma'}(t)] = [a^{\dagger}_{\mathbf{k}\sigma}(t), a_{\mathbf{k}'\sigma'}(t)]a^{\dagger}_{\mathbf{k}'\sigma'}(t) = -\delta^3_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}a^{\dagger}_{\mathbf{k}'\sigma'}(t). \quad (6.66)$$

Equation 6.66 can now be used to evaluate the quantum state $a^{\dagger}_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle$: we first consider the action of the photon number operator on this new state,

$$N_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle = \{a^{\dagger}_{\mathbf{k}\sigma}(t)N_{\mathbf{k}\sigma}(t) - [a^{\dagger}_{\mathbf{k}\sigma}(t), N_{\mathbf{k}\sigma}(t)]\}|\eta_{\mathbf{k}\sigma}\rangle, \quad (6.67)$$

and use Equation 6.66 for $\mathbf{k}' = \mathbf{k}$, and $\sigma' = \sigma$, which yields

$$N_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle = [a^{\dagger}_{\mathbf{k}\sigma}(t)N_{\mathbf{k}\sigma}(t) + a^{\dagger}_{\mathbf{k}\sigma}(t)]|\eta_{\mathbf{k}\sigma}\rangle = a^{\dagger}_{\mathbf{k}\sigma}(t)[N_{\mathbf{k}\sigma}(t) + 1]|\eta_{\mathbf{k}\sigma}\rangle. \quad (6.68)$$

Since $|\eta_{\mathbf{k}\sigma}\rangle$ is an eigenstate of the photon number operator, Equation 6.68 reduces to

$$N_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle = (n_{\mathbf{k}\sigma} + 1)a^{\dagger}_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle, \quad (6.69)$$

which clearly shows that the quantum state $a^{\dagger}_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle$ is also an eigenstate of the photon number operator, with eigenvalue $n_{\mathbf{k}\sigma} + 1$; this can be formally stated as

$$a^{\dagger}_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle = \chi_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma} + 1\rangle, \quad (6.70)$$

where $\chi_{\mathbf{k}\sigma}$ is a constant to be determined. Taking the norm of Equation 6.60, we find that

$$\langle\eta_{\mathbf{k}\sigma}|a_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle = |\chi_{\mathbf{k}\sigma}|^2\langle\eta_{\mathbf{k}\sigma} + 1|\eta_{\mathbf{k}\sigma} + 1\rangle = |\chi_{\mathbf{k}\sigma}|^2. \quad (6.71)$$

The quantum state $a_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle$ can be evaluated by using the commutator

$$[a_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}\sigma}] = a_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma} - a^{\dagger}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} = 1; \quad (6.72)$$

we then see that we can rewrite the operator $a_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma}$ as follows:

$$a_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma} = [a_{\mathbf{k}\sigma}, a^{\dagger}_{\mathbf{k}\sigma}] + a^{\dagger}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} = 1 + a^{\dagger}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} = 1 + N_{\mathbf{k}\sigma}, \quad (6.73)$$

and the sought-after quantum states are given by

$$a_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle = (1 + N_{\mathbf{k}\sigma})|\eta_{\mathbf{k}\sigma}\rangle. \quad (6.74)$$

With this, we can now evaluate the constant $\chi_{\mathbf{k}\sigma}$:

$$|\chi_{\mathbf{k}\sigma}|^2 = \langle\eta_{\mathbf{k}\sigma}|a_{\mathbf{k}\sigma}a^{\dagger}_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle = \langle\eta_{\mathbf{k}\sigma}|(1 + N_{\mathbf{k}\sigma})|\eta_{\mathbf{k}\sigma}\rangle, \quad (6.75)$$

and we know that $|\eta_{\mathbf{k}\sigma}\rangle$ is an eigenstate of $N_{\mathbf{k}\sigma}$ as presented in Equation 6.62; therefore, we have

$$|\chi_{\mathbf{k}\sigma}|^2 = (1+n_{\mathbf{k}\sigma})\langle\eta_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}\rangle = (1+n_{\mathbf{k}\sigma}), \tag{6.76}$$

which yields the important result

$$\chi_{\mathbf{k}\sigma} = \sqrt{1+n_{\mathbf{k}\sigma}}\,e^{i\theta_{\mathbf{k}\sigma}}, \tag{6.77}$$

where $\theta_{\mathbf{k}\sigma}$ is a phase factor.

It is now easy to prove by recurrence that the repeated application of the creation operator increases the photon number by one unit each time. Ignoring the phase factors, which do not contribute to the amplitude of the eigenvalue spectrum, we have

$$[a^{\dagger}_{\mathbf{k}\sigma}(t)]^m|\eta_{\mathbf{k}\sigma}\rangle = \sqrt{(1+n_{\mathbf{k}\sigma})(2+n_{\mathbf{k}\sigma})\cdots(m+n_{\mathbf{k}\sigma})}\,|\eta_{\mathbf{k}\sigma}+m\rangle, \quad m \in \mathbb{N}. \tag{6.78}$$

We now proceed in exactly the same manner with the annihilation operator acting on the eigenstate $|\eta_{\mathbf{k}\sigma}\rangle$, and consider

$$N_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle = \{a_{\mathbf{k}\sigma}(t)N_{\mathbf{k}\sigma}(t) - [a_{\mathbf{k}\sigma}(t), N_{\mathbf{k}\sigma}(t)]\}|\eta_{\mathbf{k}\sigma}\rangle. \tag{6.79}$$

We can now use the commutation rule derived in Equation 6.64, for $\mathbf{k}' = \mathbf{k}$, and $\sigma' = \sigma$, to obtain

$$[a_{\mathbf{k}\sigma}(t), N_{\mathbf{k}\sigma}(t)] = a_{\mathbf{k}\sigma}(t), \tag{6.80}$$

which, in turn, yields

$$\begin{aligned} N_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle &= [a_{\mathbf{k}\sigma}(t)N_{\mathbf{k}\sigma}(t) - a_{\mathbf{k}\sigma}(t)]|\eta_{\mathbf{k}\sigma}\rangle \\ &= a_{\mathbf{k}\sigma}(t)[N_{\mathbf{k}\sigma}(t)-1]|\eta_{\mathbf{k}\sigma}\rangle \\ &= (n_{\mathbf{k}\sigma}-1)a_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle. \end{aligned} \tag{6.81}$$

This demonstrates that $a_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle$ is an eigenstate of the number operator, with eigenvalue $\eta_{\mathbf{k}\sigma}-1$; thus, we have

$$a_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle = \kappa_{\mathbf{k}\sigma}|\eta_{\mathbf{k}\sigma}-1\rangle. \tag{6.82}$$

The normalization constant is evaluated as follows:

$$\begin{aligned} \langle\eta_{\mathbf{k}\sigma}|a^{\dagger}_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle &= |\kappa_{\mathbf{k}\sigma}|^2\langle\eta_{\mathbf{k}\sigma}-1|\eta_{\mathbf{k}\sigma}-1\rangle = |\kappa_{\mathbf{k}\sigma}|^2 \\ &= \langle\eta_{\mathbf{k}\sigma}|N_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle = n_{\mathbf{k}\sigma}, \end{aligned} \tag{6.83}$$

and

$$\kappa_{\mathbf{k}\sigma} = \sqrt{n_{\mathbf{k}\sigma}}e^{i\vartheta_{\mathbf{k}\sigma}}. \tag{6.84}$$

Furthermore, the annihilation operator can be applied repeatedly to the eigenstate $|\eta_{\mathbf{k}\sigma}\rangle$, reducing the photon number by one unit each time; this can be summarized as

$$[a_{\mathbf{k}\sigma}(t)]^m|\eta_{\mathbf{k}\sigma}\rangle = \sqrt{n_{\mathbf{k}\sigma}(n_{\mathbf{k}\sigma}-1)\cdots(n_{\mathbf{k}\sigma}-m+1)}\,|\eta_{\mathbf{k}\sigma}-m\rangle, \quad m \in \mathbb{N}. \tag{6.85}$$

Whereas the spectrum generated by the creation operator is unbounded, as shown in Equation 6.68, there is a lower limit to the annihilation spectrum. This is easily understood by considering the fact that the norm of an eigenstate must be definite positive. By examining Equation 6.83, we see that $n_{\mathbf{k}\sigma} = \langle\eta_{\mathbf{k}\sigma}|a^{\dagger}_{\mathbf{k}\sigma}(t)a_{\mathbf{k}\sigma}(t)|\eta_{\mathbf{k}\sigma}\rangle \geq 0$. The only way to guarantee a lower bound to the photon number spectrum is to define the norm of the lowest state as equal to zero:

$$\langle 0|a^{\dagger}_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}|0\rangle = 0. \tag{6.86}$$

This has an interesting consequence for the spectrum of the Hamiltonian: defining the energy operator for each electromagnetic mode, we have

$$\mathcal{H} = \sum_{\mathbf{k}}\sum_{\sigma}\hbar\omega\left[N_{\mathbf{k}\sigma}(t) + \frac{1}{2}\right] = \sum_{\mathbf{k}}\sum_{\sigma}H_{\mathbf{k}\sigma}. \tag{6.87}$$

The lowest energy level for a given mode is thus

$$\langle 0|H_{\mathbf{k}\sigma}|0\rangle = \frac{1}{2}\hbar\omega, \tag{6.88}$$

which corresponds to the vacuum energy of the mode under consideration.

6.5 Momentum of the Quantized Field

In Section 3.8, we have introduced the four-momentum of the electromagnetic fields and its relation to the Poynting vector. In this section, the same approach is modified to fit within the quantum formalism developed in this chapter.

We begin by considering the form of Maxwell's equation within the photon formalism established in the preceding sections. The temporal evolution of an operator, O, is governed by the Heisenberg equation, which states that

$$i\hbar\frac{dO}{dt} = [O,\mathcal{H}], \tag{6.89}$$

where $[O,\mathcal{H}]$ is the commutator between the operator in question and the Hamiltonian.

This is not surprising if we remember the principle of correspondence, which associates the four-gradient operator to the four-momentum: $p_\mu \to i\hbar\partial_\mu$; the time-like component of this equation shows the close relation between the energy, or the Hamiltonian, and the time-derivative operator.

The first result that can be obtained from Equation 6.89 is that the photon number operators are time-independent, since they commute with the Hamiltonian. Furthermore, the evolution of the creation and annihilation operator is governed by

$$i\frac{da_{\mathbf{k}\sigma}}{dt} = \omega a_{\mathbf{k}\sigma}, \quad i\frac{da^\dagger_{\mathbf{k}\sigma}}{dt} = -\omega a^\dagger_{\mathbf{k}\sigma}. \tag{6.90}$$

Using the definition of the electric field and magnetic induction in terms of the generalized coordinates and momenta, as expressed in Equations 6.44 and 6.45, and the relations between the creation and annihilation operators and the conjugate positions and momenta, given in Equations 6.47 and 6.48, we can derive the evolution of the electromagnetic field in the Heisenberg picture. We first have

$$\mathbb{E}(x_\mu) = \frac{1}{\sqrt{a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\sqrt{\frac{\hbar\omega}{2\varepsilon_0}}\left[ia_{\mathbf{k}\sigma}(0)\,e^{\mathbf{k}}_{\perp\sigma}e^{ik^\mu x_\mu} - ia^\dagger_{\mathbf{k}\sigma}(0)\,e^{-\mathbf{k}}_{\perp\sigma}e^{-ik^\mu x_\mu}\right], \tag{6.91}$$

for the electric field operator, $\mathbb{E}$, and

$$\mathbb{B}(x_\mu) = \frac{1}{\sqrt{a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\sqrt{\frac{\hbar}{2\omega\varepsilon_0}}\left[ia_{\mathbf{k}\sigma}(0)(\mathbf{k}\times e^{k}_{\perp\sigma})e^{ik^\mu x_\mu} - ia^\dagger_{\mathbf{k}\sigma}(0)(\mathbf{k}\times e^{-\mathbf{k}}_{\perp\sigma})e^{-ik^\mu x_\mu}\right], \tag{6.92}$$

for the magnetic induction operator, $\mathbb{B}$. Using the Heisenberg equation, we then find that

$$\begin{aligned}\nabla\times\mathbb{E}(x_\mu) + \partial_t\mathbb{B}(x_\mu) &= \mathbf{0},\\ \nabla\times\mathbb{B}(x_\mu) - \frac{1}{c^2}\partial_t\mathbb{E}(x_\mu) &= \mathbf{0},\end{aligned} \tag{6.93}$$

which are identical in form to Maxwell's equations in vacuum.

The Poynting vector, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, is closely related to the momentum of the electromagnetic field, as it essentially represents the electromagnetic momentum density; moreover, in vacuum, we have the simple relation, $\mathbf{B} = \mu_0 \mathbf{H}$, between the magnetic induction and field. Therefore, we have

$$\mathbf{G} = \int_{\mathcal{V}} \frac{\mathbf{S}}{c^2} dv = \varepsilon_0 \int_{\mathcal{V}} \mathbf{E} \times \mathbf{B}\, dv = \varepsilon_0 \iiint_{a^3} \mathbf{E} \times \mathbf{B}\, dx\, dy\, dz. \tag{6.94}$$

Here, we have used the fact that for the quantization of the electromagnetic field, a cubic cell of side a is used.

A direct generalization of $\mathbf{G}$ to a Hermitian operator, $\mathbb{G}$, using the substitution of the electric field and magnetic induction by their operator counterparts, is not possible because $\mathbb{E}$ and $\mathbb{B}$ do not commute. Therefore, a slight modification of expression 6.94 is required, and we symmetrize it by writing

$$\mathbb{G} = \frac{\varepsilon_0}{2} \iiint_{a^3} (\mathbb{E} \times \mathbb{B} - \mathbb{B} \times \mathbb{E})\, dx\, dy\, dz. \tag{6.95}$$

The electromagnetic field operators can now be introduced, as expressed in Equations 6.91 and 6.92, and we have

$$\begin{aligned} \mathbb{G} = {} & \frac{\varepsilon_0}{2} \left(\frac{1}{\sqrt{a^3}} \right)^2 \sum_{\mathbf{k}} \sum_{\sigma} \sum_{\bar{\mathbf{k}}} \sum_{\bar{\sigma}} \sqrt{\frac{\hbar \omega}{2 \varepsilon_0}} \sqrt{\frac{\hbar}{2 \bar{\omega} \varepsilon_0}} \\ & \times \iiint_{a^3} d^3x \left\{ \left[i a_{\mathbf{k}\sigma} e^{\mathbf{k}}_{\perp\sigma} e^{i k^\mu x_\mu} - i a^\dagger_{\mathbf{k}\sigma} e^{-\mathbf{k}}_{\perp\sigma} e^{-i k^\mu x_\mu} \right] \right. \\ & \left. \times \left[i a_{\bar{\mathbf{k}}\bar{\sigma}} (\bar{\mathbf{k}} \times e^{\bar{\mathbf{k}}}_{\perp\bar{\sigma}}) e^{i \bar{k}^\mu x_\mu} - i a^\dagger_{\bar{\mathbf{k}}\bar{\sigma}} (\bar{\mathbf{k}} \times e^{-\bar{\mathbf{k}}}_{\perp\bar{\sigma}}) e^{-i \bar{k}^\mu x_\mu} \right] + \dagger \right\}. \end{aligned} \tag{6.96}$$

In Equation 6.96, the symbols $+\dagger$ indicate that the Hermitian conjugate must be added to the original expression. Because of the orthogonality of the photon modes, the volume integrals yield $a^3 \delta^3_{\pm\mathbf{k}\bar{\mathbf{k}}}$, and the sum over the wavenumber is diagonalized. We now have

$$\mathbb{G} = \frac{\hbar}{4} \sum_{\mathbf{k}} \sum_{\sigma} \sum_{\bar{\sigma}} \{ [i a_{\mathbf{k}\sigma} e^{\mathbf{k}}_{\perp\sigma} - i a^\dagger_{\mathbf{k}\sigma} e^{-\mathbf{k}}_{\perp\sigma}] \times [i a_{\mathbf{k}\bar{\sigma}} (\mathbf{k} \times e^{\mathbf{k}}_{\perp\bar{\sigma}}) - i a^\dagger_{\mathbf{k}\bar{\sigma}} (\mathbf{k} \times e^{-\mathbf{k}}_{\perp\bar{\sigma}})] + \dagger \}. \tag{6.97}$$

The double cross-products are

$$\begin{aligned} & e^{\mathbf{k}}_{\perp\sigma} \times (\mathbf{k} \times e^{\mathbf{k}}_{\perp\bar{\sigma}}), \quad e^{\mathbf{k}}_{\perp\sigma} \times (\mathbf{k} \times e^{-\mathbf{k}}_{\perp\bar{\sigma}}) \\ & e^{-\mathbf{k}}_{\perp\sigma} \times (\mathbf{k} \times e^{\mathbf{k}}_{\perp\bar{\sigma}}), \quad e^{-\mathbf{k}}_{\perp\sigma} \times (\mathbf{k} \times e^{-\mathbf{k}}_{\perp\bar{\sigma}}). \end{aligned} \tag{6.98}$$

Using the formula, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, we have

$$e_{\perp\sigma}^{\mathbf{k}} \times (\mathbf{k} \times e_{\perp\bar{\sigma}}^{\mathbf{k}}) = (e_{\perp\sigma}^{\mathbf{k}} \cdot e_{\perp\bar{\sigma}}^{\mathbf{k}})\mathbf{k} - (e_{\perp\sigma}^{\mathbf{k}} \cdot \mathbf{k})e_{\perp\bar{\sigma}}^{\mathbf{k}} = \delta_{\sigma\bar{\sigma}}\mathbf{k}, \tag{6.99}$$

where the last equality holds because the polarization vector and wavenumber are orthogonal for real photons: $e_{\perp\sigma}^{\mathbf{k}} \cdot \mathbf{k} = 0$. The Kronecker symbol simply reflects the fact that the polarization vectors are orthogonal and of unit length. The double sum over polarization states is now reduced to a single sum, and we finally obtain

$$\mathbb{G} = \frac{\hbar}{2}\sum_{\mathbf{k}}\sum_{\sigma}(a_{\mathbf{k}\sigma}^{\dagger}a_{\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}^{\dagger})\mathbf{k}, \tag{6.100}$$

where we have used the fact that the terms $a_{\mathbf{k}\sigma}a_{-\mathbf{k}\sigma}$ and $a_{\mathbf{k}\sigma}^{\dagger}a_{-\mathbf{k}\sigma}^{\dagger}$ are antisymmetrical.

This important result can be compared with Equation 6.55 for the energy, and we see that we can group the energy and momentum of the quantized field in a single four-operator:

$$\mathbb{G}_{\mu} = \frac{1}{2}\sum_{\mathbf{k}}\sum_{\sigma}(a_{\mathbf{k}\sigma}^{\dagger}a_{\mathbf{k}\sigma} + a_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}^{\dagger})\hbar k_{\mu}. \tag{6.101}$$

The transformation properties of this operator under the Lorentz group are identical to that of the four-wavenumber, as the photon number and polarization state must be independent of the observation frame; in other words, photons cannot be created or annihilated, nor can their polarization be flipped by switching reference frame.

6.6 Angular Momentum of the Quantized Field

Proceeding in the same manner as for the momentum, we begin with classical theory and define the angular momentum of the electromagnetic field as follows:

$$\mathbf{J}(\mathbf{r}) = \int_{\mathcal{V}}(\mathbf{x} - \mathbf{r}) \times \frac{\mathbf{S}}{c^2}d^3x = \varepsilon_0\int_{\mathcal{V}}(\mathbf{x} - \mathbf{r}) \times [\mathbf{E}(x_{\mu}) \times \mathbf{B}(x_{\mu})]\,d^3x. \tag{6.102}$$

It is immediately seen that we can decompose the angular momentum into two components,

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}(\mathbf{0}) - \mathbf{r} \times \mathbf{G}, \tag{6.103}$$

where **G** is the electromagnetic field momentum discussed in the previous section.

We will return on the classical intrinsic angular momentum of the field, **J**(**0**), in the next section. Here, we follow the procedure outlined in the preceding section; in particular, we symmetrize the operator to guarantee that it is Hermitian:

$$\mathbb{J}(\mathbf{r}) = \frac{\varepsilon_0}{2}\iiint_{a^3}(\mathbf{x}-\mathbf{r})\times(\mathbb{E}\times\mathbb{B}-\mathbb{B}\times\mathbb{E})d^3x = \mathbb{J}(\mathbf{0})-\mathbf{r}\times\mathbb{G}. \tag{6.104}$$

Thus far, we have not shown that this operator does not explicitly depend on time. The fact that for the four-momentum operator we have $\partial_t\mathbb{G}_\mu = 0$ is borne out by the commutation of this operator with the Hamiltonian; therefore, the question reduces to that of the time-independence of the intrinsic angular momentum of the electromagnetic field.

To perform this demonstration, we return to the Heisenberg equations for the field operators, as expressed in Equation 6.93. We then have

$$\begin{aligned}\frac{d\mathbb{J}(\mathbf{0})}{dt} &= \frac{\varepsilon_0}{2}\iiint_{a^3}\mathbf{x}\times\left[\frac{\partial}{\partial t}(\mathbb{E}\times\mathbb{B}-\mathbb{B}\times\mathbb{E})\right]d^3x\\ &= \frac{\varepsilon_0}{2}\iiint_{a^3}\mathbf{x}\times\left[\frac{\partial\mathbb{E}}{\partial t}\times\mathbb{B}+\mathbb{E}\times\frac{\partial\mathbb{B}}{\partial t}-\frac{\partial\mathbb{B}}{\partial t}\times\mathbb{E}-\mathbb{B}\times\frac{\partial\mathbb{E}}{\partial t}\right]d^3x\\ &= \frac{\varepsilon_0}{2}\iiint_{a^3}\mathbf{x}\times[c^2(\nabla\times\mathbb{B})\times\mathbb{B}-\mathbb{E}\times(\nabla\times\mathbb{E})\\ &\quad+(\nabla\times\mathbb{E})\times\mathbb{E}-\mathbb{B}\times c^2(\nabla\times\mathbb{B})]\,d^3x\\ &= \iiint_{a^3}\mathbf{x}\times[\mu_0^{-1}(\nabla\times\mathbb{B})\times\mathbb{B}+\varepsilon_0(\nabla\times\mathbb{E})\times\mathbb{E}]\,d^3x,\end{aligned} \tag{6.105}$$

where we have used the fact that $\varepsilon_0\mu_0c^2 = 1$, the antisymmetrical nature of the cross-product operator, with $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, and the fact that the field operators commute with each other at a fixed time: $[\mathbb{F}_i(\mathbf{x},t),\mathbb{F}_j(\bar{\mathbf{x}},t)] = 0$, where $\mathbb{F}$ represents either $\mathbb{E}$ or $\mathbb{B}$.

Next, we have

$$\begin{aligned}\nabla(\mathbf{a}\cdot\mathbf{b}) &= (\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a}),\\ \nabla(\mathbf{a}^2) &= 2[(\mathbf{a}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{a})],\\ \mathbf{a}\times(\nabla\times\mathbf{a}) &= \frac{1}{2}\nabla(\mathbf{a}^2)-(\mathbf{a}\cdot\nabla)\mathbf{a}.\end{aligned} \tag{6.106}$$

Identifying $\mathbb{B}\equiv\mathbf{a}$, we find that

$$(\nabla\times\mathbb{B})\times\mathbb{B} = (\mathbb{B}\cdot\nabla)\mathbb{B}-\frac{1}{2}\nabla(\mathbb{B}\cdot\mathbb{B}). \tag{6.107}$$

Taking the cross-product of Equation 6.107 with the position vector, **x**, we then have

$$\mathbf{x}\times[(\nabla\times\mathbb{B})\times\mathbb{B}] = \mathbf{x}\times\left[(\mathbb{B}\cdot\nabla)\mathbb{B}-\frac{1}{2}\nabla(\mathbb{B}\cdot\mathbb{B})\right]. \tag{6.108}$$

To simplify this expression, we first show that

$$-\frac{1}{2}\mathbf{x}\times[\nabla(\mathbb{B}\cdot\mathbb{B})] = \frac{1}{2}\nabla\times[\mathbf{x}(\mathbb{B}\cdot\mathbb{B})]. \tag{6.109}$$

We start with

$$\nabla\times(f\mathbf{u}) = f(\nabla\times\mathbf{u})-\mathbf{u}\times\nabla f, \tag{6.110}$$

as can be seen by considering a given component of $\nabla\times(f\mathbf{u})$:

$$\begin{aligned}
[\nabla\times(f\mathbf{u})]_i &= \partial_j(fu_k)-\partial_k(fu_j)\\
&= u_k\partial_j f+f\partial_j u_k-u_j\partial_k f-f\partial_k u_j\\
&= u_k\partial_j f-u_j\partial_k f+f(\partial_j u_k-\partial_k u_j)\\
&= -(\mathbf{u}\times\nabla f)_i+f(\nabla\times\mathbf{u})_i.
\end{aligned} \tag{6.111}$$

We can then use Equation 6.110 to write

$$\begin{aligned}
\nabla\times[\mathbf{x}(\mathbb{B}\cdot\mathbb{B})] &= \nabla\times(\mathbb{B}^2\mathbf{x})\\
&= \mathbb{B}^2(\nabla\times\mathbf{x})-\mathbf{x}\times\nabla\mathbb{B}^2\\
&= -\mathbf{x}\times\nabla\mathbb{B}^2,
\end{aligned} \tag{6.112}$$

because $\nabla\times\mathbf{x} = \mathbf{0}$.

Multiplying Equation 6.112 by 1/2 yields the desired result.

Next, we consider the term $\mathbf{x}\times[(\mathbb{B}\cdot\nabla)\mathbb{B}]$. In tensorial form, we will show that

$$\mathbf{x}\times[(\mathbb{B}\cdot\nabla)\mathbb{B}] = \nabla\cdot[\mathbb{B}\otimes(\mathbf{x}\times\mathbb{B})], \tag{6.113}$$

where the symbol $\otimes$ denotes the tensorial product:

$$\mathbf{u}\otimes\mathbf{v} = \mathbb{T},\quad T_{ij} = u_i v_j. \tag{6.114}$$

Using Einstein's convention, we can express the three-divergence of the tensorial product as follows:

$$\partial_i T_{ij} = \partial_i(u_i v_j) = v_j(\partial_i u_i)+(u_i\partial_i)v_j, \tag{6.115}$$

or

$$\nabla \cdot (\mathbf{u} \otimes \mathbf{v}) = \mathbf{v}(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{v}. \tag{6.116}$$

Identifying $\mathbf{u} \equiv \mathbb{B}$, and $\mathbf{v} \equiv \mathbf{x} \times \mathbb{B}$, we then find that

$$\nabla \cdot [\mathbb{B} \otimes (\mathbf{x} \times \mathbb{B})] = (\mathbf{x} \times \mathbb{B})(\nabla \cdot \mathbb{B}) + (\mathbb{B} \cdot \nabla)(\mathbf{x} \times \mathbb{B}). \tag{6.117}$$

As the divergence of the magnetic induction is always equal to zero, we have to show that

$$\nabla \cdot [\mathbb{B} \otimes (\mathbf{x} \times \mathbb{B})] = (\mathbb{B} \cdot \nabla)(\mathbf{x} \times \mathbb{B}) = \mathbf{x} \times [(\mathbb{B} \cdot \nabla)\mathbb{B}]. \tag{6.118}$$

This last equality is easily demonstrated in coordinate form:

$$\begin{aligned}
[(\mathbb{B} \cdot \nabla)(\mathbf{x} \times \mathbb{B})]_i &= (\mathbb{B} \cdot \nabla)[\mathbf{x} \times \mathbb{B}]_i \\
&= (\mathbb{B} \cdot \nabla)(x_j \mathbb{B}_k - x_k \mathbb{B}_j) \\
&= [x_j(\mathbb{B} \cdot \nabla)\mathbb{B}_k - x_k(\mathbb{B} \cdot \nabla)\mathbb{B}_j] + \mathbb{B}_k(\mathbb{B} \cdot \nabla)x_j - \mathbb{B}_j(\mathbb{B} \cdot \nabla)x_k \\
&= \{\mathbf{x} \times [(\mathbb{B} \cdot \nabla)\mathbb{B}]\}_i + \mathbb{B}_k(\mathbb{B} \cdot \nabla)x_j - \mathbb{B}_j(\mathbb{B} \cdot \nabla)x_k \\
&= \{\mathbf{x} \times [(\mathbb{B} \cdot \nabla)\mathbb{B}]\}_i + \mathbb{B}_k(\mathbb{B}_i \partial_i)x_j - \mathbb{B}_j(\mathbb{B}_i \partial_i)x_k \\
&= \{\mathbf{x} \times [(\mathbb{B} \cdot \nabla)\mathbb{B}]\}_i + \mathbb{B}_k \mathbb{B}_i \delta_{ij} - \mathbb{B}_j \mathbb{B}_i \delta_{ik} \\
&= \{\mathbf{x} \times [(\mathbb{B} \cdot \nabla)\mathbb{B}]\}_i + \mathbb{B}_k \mathbb{B}_j - \mathbb{B}_j \mathbb{B}_k \\
&= \{\mathbf{x} \times [(\mathbb{B} \cdot \nabla)\mathbb{B}]\}_i.
\end{aligned} \tag{6.119}$$

Here, we have used the relation $\partial_i x_j = \delta_{ij}$.

Therefore, we have shown that

$$\mathbf{x} \times [(\nabla \times \mathbb{B}) \times \mathbb{B}] = -\frac{1}{2}\nabla \times [\mathbf{x}(\mathbb{B} \cdot \mathbb{B})] - \nabla \cdot [\mathbb{B} \otimes (\mathbf{x} \times \mathbb{B})]. \tag{6.120}$$

A similar relation can be derived for the electric field operator:

$$\mathbf{x} \times [(\nabla \times \mathbb{E}) \times \mathbb{E}] = -\frac{1}{2}\nabla \times [\mathbf{x}(\mathbb{E} \cdot \mathbb{E})] - \nabla \cdot [\mathbb{E} \otimes (\mathbf{x} \times \mathbb{E})]. \tag{6.121}$$

Note, however, that for the electric field, the equivalent to Equation 6.118 works because the absence of charges in vacuum yields $\nabla \cdot \mathbb{E} = 0$.

Having expressed the integrand in Equation 6.105 in terms of a curl and a divergence, we can transform the volume integral into a surface integral by applying the divergence, or Gauss, theorem:

$$\begin{aligned}\int_{\mathcal{V}}(\nabla\cdot\mathbf{A})\,d^3x &= \int_S(\mathbf{n}\cdot\mathbf{A})\,ds,\\ \int_{\mathcal{V}}(\nabla\times\mathbf{A})\,d^3x &= \int_S(\mathbf{n}\times\mathbf{A})\,ds.\end{aligned} \tag{6.122}$$

In our case, we find

$$\begin{aligned}\frac{d\mathbb{J}(\mathbf{0})}{dt} &= \frac{1}{2}\int_S[\mathbf{n}\times\mathbf{x}(\mu_0^{-1}\mathbb{B}^2+\varepsilon_0\mathbb{E}^2)]\,ds\\ &\quad+\int_S\{\mathbf{n}\cdot[\mu_0^{-1}\mathbb{B}\otimes(\mathbf{x}\times\mathbb{B})+\varepsilon_0\mathbb{E}\otimes(\mathbf{x}\times\mathbb{E})]\}\,ds,\end{aligned} \tag{6.123}$$

where the surface corresponds to that of the $a\times a\times a$ cube used to quantize the free electromagnetic field.

The periodic boundary conditions on each surface of the cube imply that the fields have the same values on opposite surfaces, whereas the vectors $\mathbf{n}\times\mathbf{x}$ are opposite; thus, the first integral vanishes, and Equation 6.123 reduces to

$$\begin{aligned}\frac{d\mathbb{J}(\mathbf{0})}{dt} &= \int_S\{\mathbf{n}\cdot[\mu_0^{-1}\mathbb{B}\otimes(\mathbf{x}\times\mathbb{B})+\varepsilon_0\mathbb{E}\otimes(\mathbf{x}\times\mathbb{E})]\}\,ds\\ &= \int_S[\mu_0^{-1}(\mathbf{n}\cdot\mathbb{B})(\mathbf{x}\times\mathbb{B})+\varepsilon_0(\mathbf{n}\cdot\mathbb{E})(\mathbf{x}\times\mathbb{E})]\,ds.\end{aligned} \tag{6.124}$$

The second equality in Equation 6.124 derives from the fact that

$$n_iT_{ij} = n_i(u_iv_j) = (n_iu_i)v_j, \tag{6.125}$$

or

$$\mathbf{n}\cdot\mathbb{T} = \mathbf{n}\cdot(\mathbf{u}\otimes\mathbf{v}) = (\mathbf{n}\cdot\mathbf{u})\mathbf{v}. \tag{6.126}$$

Equation 6.124 can also be written in component form:

$$\begin{aligned}\frac{d\mathbb{J}_i(\mathbf{0})}{dt} &= \int_S[\mu_0^{-1}(n_l\mathbb{B}_l)(x_j\mathbb{B}_k-x_k\mathbb{B}_j)+\varepsilon_0(n_l\mathbb{E}_l)(x_j\mathbb{E}_k-x_k\mathbb{E}_j)]\,ds\\ &= \varepsilon_{ijk}\int_S x_jn_l(\mu_0^{-1}\mathbb{B}_k\mathbb{B}_l+\varepsilon_0\mathbb{E}_k\mathbb{E}_l)\,ds,\end{aligned} \tag{6.127}$$

where ε_{ijk} is the completely antisymmetrical Levi–Civita tensor.

Using Equations 6.91 and 6.92, the components of the electric field and magnetic induction operators can be expressed in terms of creation and annihilation operators. For the electric field, we have

$$\begin{aligned}\mathbb{E}_i(x_\mu) &= \frac{1}{\sqrt{a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\sqrt{\frac{\hbar\omega}{2\varepsilon_0}}\Big[ia_{\mathbf{k}\sigma}(0)e^{\mathbf{k}}_{\perp\sigma}e^{ik^\mu x_\mu}-ia^\dagger_{\mathbf{k}\sigma}(0)e^{-\mathbf{k}}_{\perp\sigma}e^{-ik^\mu x_\mu}\Big]_i\\ &= \mathbb{E}_i^+(x_\mu)+\mathbb{E}_i^-(x_\mu),\end{aligned} \tag{6.128}$$

where we have introduced

$$\begin{aligned} \mathbb{E}_i^+(x_\mu) &= \frac{1}{\sqrt{a^3}} \sum_{\mathbf{k}} \sum_{\sigma} \sqrt{\frac{\hbar\omega}{2\varepsilon_0}} \left[i a_{\mathbf{k}\sigma}(0) e_{\perp\sigma}^{\mathbf{k}} e^{ik^\mu x_\mu} \right]_i, \\ \mathbb{E}_i^-(x_\mu) &= \frac{1}{\sqrt{a^3}} \sum_{\mathbf{k}} \sum_{\sigma} \sqrt{\frac{\hbar\omega}{2\varepsilon_0}} \left[-i a_{\mathbf{k}\sigma}^\dagger(0) e_{\perp\sigma}^{-\mathbf{k}} e^{-ik^\mu x_\mu} \right]_i. \end{aligned} \tag{6.129}$$

Similar expressions can be defined for the magnetic induction operator.

We can now write

$$\begin{aligned} \mathbb{E}_i(x_\mu)\mathbb{E}_j(x_\mu) &= [\mathbb{E}_i^+(x_\mu) + \mathbb{E}_i^-(x_\mu)][\mathbb{E}_j^+(x_\mu) + \mathbb{E}_j^-(x_\mu)] \\ &= \mathbb{E}_i^+(x_\mu)\mathbb{E}_j^+(x_\mu) + \mathbb{E}_i^+(x_\mu)\mathbb{E}_j^-(x_\mu) \\ &\quad + \mathbb{E}_i^-(x_\mu)\mathbb{E}_j^+(x_\mu) + \mathbb{E}_i^-(x_\mu)\mathbb{E}_j^-(x_\mu). \end{aligned} \tag{6.130}$$

Before considering the order of the operators in Equation 6.130, which is important, a few words about the terminology used here will be useful. The plus and minus signs refer to positive and negative frequencies, respectively; however, the operator labeled with a plus sign corresponds to annihilation operators only, while its counterpart is composed entirely of creation operators. We also note that in the so-called normal or Weyl ordering of operators, creation operators must appear to the left of annihilation operators. In other words, photons must first be created, before annihilation occurs. We see that in Equation 6.130, the only term that is not normally ordered is $\mathbb{E}_i^+\mathbb{E}_j^-$; however, because $\mathbb{E}_i^+$ and $\mathbb{E}_j^-$ commute, the order can be reversed so that Equation 6.130 is now entirely in normal order:

$$\begin{aligned} \mathbb{E}_i(x_\mu)\mathbb{E}_j(x_\mu) &= \mathbb{E}_i^+(x_\mu)\mathbb{E}_j^+(x_\mu) + \mathbb{E}_j^-(x_\mu)\mathbb{E}_i^+(x_\mu) \\ &\quad + \mathbb{E}_i^-(x_\mu)\mathbb{E}_j^+(x_\mu) + \mathbb{E}_i^-(x_\mu)\mathbb{E}_j^-(x_\mu). \end{aligned} \tag{6.131}$$

At this point, a physical argument can be used to reduce the integral remaining in Equation 6.127: one must consider the limit where the cubic volume used to quantize the free electromagnetic field goes to infinity as a^3. In this case, the normally ordered operators in Equation 6.131 have a vanishingly small expectation value for photons localized away from the boundary at infinity, and the angular momentum is conserved.

6.7 Classical Spin of the Electromagnetic Field

We now return to the intrinsic angular momentum of the electromagnetic field in vacuum, which is related to the spin of photons. In the classical approach to this problem, the magnetic induction is expressed in terms of the scalar potential, so that

$$\mathbf{E} \times \mathbf{B} = \mathbf{E} \times (\nabla \times \mathbf{A}). \tag{6.132}$$

Furthermore, the gauge condition is chosen so that $\phi = 0$ and $\nabla \cdot \mathbf{A} = 0$; we then have

$$\mathbf{E} = -\partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{6.133}$$

or, in terms of components,

$$E_i = -\partial_t A_i, \quad B_i = \partial_j A_k - \partial_k A_j = \varepsilon_{ijk} \partial_j A_k. \tag{6.134}$$

The intrinsic angular momentum is then given by

$$\mathbf{J} = \int_{\mathcal{V}} \frac{\mathbf{x} \times \mathbf{S}}{c^2} d^3x = \varepsilon_0 \int_{\mathcal{V}} \mathbf{x} \times (\mathbf{E} \times \mathbf{B})\, d^3x, \tag{6.135}$$

which translates to

$$\begin{aligned} J_i &= \varepsilon_0 \varepsilon_{ijk} \int_{\mathcal{V}} x_j (\mathbf{E} \times \mathbf{B})_k \, d^3x \\ &= \varepsilon_0 \varepsilon_{ijk} \int_{\mathcal{V}} x_j \varepsilon_{klm} E_l B_m \, d^3x \\ &= \varepsilon_0 \varepsilon_{ijk} \varepsilon_{klm} \int_{\mathcal{V}} x_j (-\partial_t A_l) B_m \, d^3x \\ &= -\varepsilon_0 \varepsilon_{ijk} \varepsilon_{klm} \int_{\mathcal{V}} x_j (\partial_t A_l) \varepsilon_{mnp} (\partial_n A_p) \, d^3x \\ &= -\varepsilon_0 \varepsilon_{ijk} \varepsilon_{klm} \varepsilon_{mnp} \int_{\mathcal{V}} x_j (\partial_t A_l)(\partial_n A_p) \, d^3x, \end{aligned} \tag{6.136}$$

as expressed in terms of coordinates. In order to apply the divergence theorem, as stated in Equation 6.122, the integrand is recast as follows:

$$x_j (\partial_t A_l)(\partial_n A_p) = \partial_n [x_j (\partial_t A_l) A_p] - (\partial_n x_j)(\partial_t A_l) A_p - x_j A_p [\partial_n (\partial_t A_l)]. \tag{6.137}$$

The quantity $\partial_n x_j = \delta_{nj}$, and the surface integral vanishes, so we are left with

$$J_i = \varepsilon_0 \varepsilon_{ijk} \varepsilon_{klm} \varepsilon_{mnp} \int_{\mathcal{V}} \{ \delta_{nj} (\partial_t A_l) A_p + x_j A_p [\partial_n (\partial_t A_l)] \} \, d^3x. \tag{6.138}$$

We now make use of the contraction formula,

$$\varepsilon_{klm} \varepsilon_{mnp} = \delta_{kn} \delta_{lp} - \delta_{kp} \delta_{ln}, \tag{6.139}$$

to reduce Equation 6.138 to

$$\begin{aligned} J_i &= \varepsilon_0 \varepsilon_{ijk}(\delta_{kn}\delta_{lp} - \delta_{kp}\delta_{ln})\int_{\mathcal{V}} \{\delta_{nj}(\partial_t A_l)A_p + x_j A_p[\partial_n(\partial_t A_l)]\}\, d^3x \\ &= \varepsilon_0 \varepsilon_{ijk}\int_{\mathcal{V}} (\delta_{kn}\delta_{lp} - \delta_{kp}\delta_{ln})\{\delta_{nj}(\partial_t A_l)A_p + x_j A_p[\partial_n(\partial_t A_l)]\}\, d^3x \\ &= \varepsilon_0 \varepsilon_{ijk}\int_{\mathcal{V}} [\delta_{jk}(\partial_t A_l)A_l - (\partial_t A_j)A_k + x_j A_l \partial_k(\partial_t A_l) - x_j A_k \partial_l(\partial_t A_l)]\, d^3x. \end{aligned} \tag{6.140}$$

The term $\partial_l(\partial_t A_l)$ is equal to zero:

$$\partial_l(\partial_t A_l) = \frac{\partial}{\partial x_l}\frac{\partial A_l}{\partial t} = \frac{\partial}{\partial t}\frac{\partial A_l}{\partial x_l} = \partial_t(\nabla \cdot \mathbf{A}) = 0. \tag{6.141}$$

Moreover, we have

$$\varepsilon_{ijk}\delta_{jk} = \varepsilon_{ijj} = 0. \tag{6.142}$$

As a result, Equation 6.140 further simplifies to read

$$J_i = \varepsilon_0 \varepsilon_{ijk}\int_{\mathcal{V}} [x_j A_l \partial_k(\partial_t A_l) - (\partial_t A_j)A_k]\, d^3x. \tag{6.143}$$

The first term can be related to the angular momentum operator,

$$\mathbb{L}_i = i\hbar \varepsilon_{ijk} x_j \partial_k, \tag{6.144}$$

and depends on the reference frame because of the term x_j. On the other hand, the term

$$\begin{aligned} S_i &= -\varepsilon_0 \varepsilon_{ijk}\int_{\mathcal{V}} (\partial_t A_j)A_k\, d^3x, \\ \mathbf{S} &= \varepsilon_0 \int_{\mathcal{V}} \left(\mathbf{A} \times \frac{\partial \mathbf{A}}{\partial t}\right) d^3x = \varepsilon_0 \int_{\mathcal{V}} (\mathbf{E} \times \mathbf{A})\, d^3x, \end{aligned} \tag{6.145}$$

is frame-independent and corresponds to the spin of the free electromagnetic field. For a circularly polarized plane wave, we have

$$\mathbf{A} = A_0[\hat{x}\cos(k_\mu x^\mu) \pm \hat{y}\sin(k_\mu x^\mu)], \tag{6.146}$$

and

$$\frac{\partial \mathbf{A}}{\partial t} = c\frac{\partial \mathbf{A}}{\partial x_0} = A_0(-ck_0)[-\hat{x}\sin(k_\mu x^\mu) \pm \hat{y}\cos(k_\mu x^\mu)]. \tag{6.147}$$

The corresponding spin density is

$$\begin{aligned}\frac{d\mathbf{S}}{d\mathcal{V}} &= \varepsilon_0\left(\mathbf{A}\times\frac{\partial\mathbf{A}}{\partial t}\right)\\ &= ck_0\varepsilon_0A_0^2[\hat{x}\cos(k_\mu x^\mu)\pm\hat{y}\sin(k_\mu x^\mu)]\times[\hat{x}\sin(k_\mu x^\mu)\mp\hat{y}\cos(k_\mu x^\mu)]\\ &= ck_0\varepsilon_0A_0^2\{(\hat{x}\times\hat{y})[\mp\cos^2(k_\mu x^\mu)]+(\hat{y}\times\hat{x})[\pm\sin^2(k_\mu x^\mu)]\}\\ &= \mp\omega\varepsilon_0A_0^2\hat{z}.\end{aligned} \tag{6.148}$$

6.8 Photon Spin

Using Equation 6.145 and the symmetrization technique described for the field momentum operator, we have

$$\mathbb{S} = \frac{\varepsilon_0}{2}\iiint_{a^3}(\mathbb{E}\times\mathbb{A}-\mathbb{A}\times\mathbb{E})\,d^3x, \tag{6.149}$$

and we can replace the electric field and vector potential operators by their expansions, as expressed in Equation 6.91 and

$$\mathbb{A}(x_\mu) = \frac{1}{\sqrt{a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\sqrt{\frac{\hbar}{2\varepsilon_0\omega}}\left[a_{\mathbf{k}\sigma}e^{\mathbf{k}}_{\perp\sigma}e^{ik^\mu x_\mu}+a^\dagger_{\mathbf{k}\sigma}e^{-\mathbf{k}}_{\perp\sigma}e^{-ik^\mu x_\mu}\right]. \tag{6.150}$$

With this, we have

$$\begin{aligned}\mathbb{S} = \frac{\hbar}{4a^3}\sum_{\mathbf{k},\sigma}\sum_{\bar{\mathbf{k}},\bar{\sigma}}\sqrt{\frac{\omega}{\bar{\omega}}}\iiint_{a^3}\Big\{&\left[ia_{\mathbf{k}\sigma}e^{\mathbf{k}}_{\perp\sigma}e^{ik^\mu x_\mu}-ia^\dagger_{\mathbf{k}\sigma}e^{-\mathbf{k}}_{\perp\sigma}e^{-ik^\mu x_\mu}\right]\\ &\times\left[a_{\bar{\mathbf{k}}\bar{\sigma}}e^{\bar{\mathbf{k}}}_{\perp\bar{\sigma}}e^{i\bar{k}^\mu x_\mu}+a^\dagger_{\bar{\mathbf{k}}\bar{\sigma}}e^{-\bar{\mathbf{k}}}_{\perp\bar{\sigma}}e^{-i\bar{k}^\mu x_\mu}\right]+\dagger\Big\}d^3x.\end{aligned} \tag{6.151}$$

The reduction of the volume integral is identical to that used for the field momentum operator, and the commutation relations for the creation and annihilation operators, as expressed in Equation 6.49, can be used to obtain

$$\mathbb{S} = i\hbar\sum_{\mathbf{k},\sigma}\sum_{\bar{\sigma}}\left(a^\dagger_{\mathbf{k}\bar{\sigma}}a_{\mathbf{k}\sigma}+\frac{1}{2}\delta_{\sigma\bar{\sigma}}\right)(e^{\mathbf{k}}_{\perp\sigma}\times e^{-\mathbf{k}}_{\perp\bar{\sigma}}). \tag{6.152}$$

Further simplification can be achieved by an appropriate projection basis for the polarization states. In particular, for linear polarization, we have

$$e^{\mathbf{k}}_{\perp\sigma}\times e^{-\mathbf{k}}_{\perp\bar{\sigma}} = \pm\frac{\mathbf{k}}{|\mathbf{k}|}(1-\delta_{\sigma\bar{\sigma}}), \tag{6.153}$$

and we find that

$$\mathbb{S} = i\hbar \sum_{\mathbf{k},\sigma} \sum_{\bar{\sigma}} \left(a^{\dagger}_{\mathbf{k}\bar{\sigma}} a_{\mathbf{k}\sigma} + \frac{1}{2}\delta_{\sigma\bar{\sigma}} \right) \left[\pm \frac{\mathbf{k}}{|\mathbf{k}|}(1 - \delta_{\sigma\bar{\sigma}}) \right]$$

$$= i \sum_{\mathbf{k}} \frac{\hbar \mathbf{k}}{|\mathbf{k}|} (a^{\dagger}_{\mathbf{k}\sigma 2} a_{\mathbf{k}1} - a^{\dagger}_{\mathbf{k}\sigma 1} a_{\mathbf{k}2}). \tag{6.154}$$

We recover the classical result derived in Section 6.7: the spin is in the direction of propagation of the wave.

6.9 Vacuum Fluctuations

In this section, we give a short overview of the well-known question of vacuum fluctuations of the free quantized electromagnetic field. For considerably more detailed descriptions, we refer the reader to the textbooks by Mandel and Wolf, Loudon, Dirac, and Pauli, which are listed in the references to this chapter.

The key idea is that the lowest energy level for photons, corresponding to the vacuum state, has both a nonzero energy eigenvalue and nonzero fluctuations. In turn, this physical fact is at the origin of some of the divergence problems encountered in QED before the renormalization program was completed by Feynman, Schwinger, Tomonaga, and Dyson.

In the following discussion, the vacuum state will be labeled by the bra and kets $\langle 0|$ and $|0\rangle$, respectively. As discussed in Section 6.4, the expectation value of the creation and annihilation operators is zero for the vacuum state:

$$\langle 0 | a^{\dagger}_{\mathbf{k}\sigma} = 0 = a_{\mathbf{k}\sigma} | 0 \rangle; \tag{6.155}$$

on the other hand, the energy eigenvalue of the vacuum state is

$$\langle 0 | H_{\mathbf{k}\sigma} | 0 \rangle = \frac{1}{2}\hbar\omega, \tag{6.156}$$

as shown in Section 6.4.

We now turn our attention to the expectation value of the field operator in the vacuum state. As discussed earlier, any given field operator can be written in terms of creation and annihilation operators, with

$$\mathbb{F}(x_\mu) = \frac{1}{\sqrt{a^3}} \sum_{\mathbf{k}} \sum_{\sigma} \left[\mathcal{F}(k_\mu) a_{\mathbf{k}\sigma} e^{\mathbf{k}}_{\perp\sigma} e^{ik^\mu x_\mu} + \dagger \right], \tag{6.157}$$

which reduces to

$$\mathbb{F}(x_\mu) = \frac{1}{\sqrt{a^3}}\sum_{\mathbf{k}}\sum_{\sigma}\Big[\mathcal{F}(\omega)a_{\mathbf{k}\sigma}e^{\mathbf{k}}_{\perp\sigma}e^{ik^\mu x_\mu} + \dagger\Big], \tag{6.158}$$

because of the mass-shell constraint $k_\mu k^\mu = \mathbf{k}^2 - (\omega/c)^2 = 0$ for photons.

We can now use Equations 6.155 and 6.158 to show that the vacuum expectation value for any field operator $\mathbb{F}$ is zero:

$$\langle 0|\mathbb{F}(x_\mu)|0\rangle = 0 \tag{6.159}$$

The vacuum fluctuations for the field operator $\mathbb{F}$ are defined as

$$\Delta\langle\mathbb{F}\rangle = \sqrt{|\langle 0|\mathbb{F}^2(x_\mu)|0\rangle - [\langle 0|\mathbb{F}(x_\mu)|0\rangle]^2|} = \sqrt{\langle 0|\mathbb{F}^2(x_\mu)|0\rangle}. \tag{6.160}$$

Therefore, we need to evaluate the vacuum expectation for the square of the field operator, namely, $\langle 0|\mathbb{F}^2(x_\mu)|0\rangle$. This quantity will involve four different expectations values:

$$\begin{aligned}\langle 0|a_{\mathbf{k}\sigma}a_{\bar{\mathbf{k}}\bar{\sigma}}|0\rangle &= 0,\\ \langle 0|a_{\mathbf{k}\sigma}a^\dagger_{\bar{\mathbf{k}}\bar{\sigma}}|0\rangle &\neq 0,\\ \langle 0|a^\dagger_{\mathbf{k}\sigma}a_{\bar{\mathbf{k}}\bar{\sigma}}|0\rangle &= 0,\\ \langle 0|a^\dagger_{\mathbf{k}\sigma}a^\dagger_{\bar{\mathbf{k}}\bar{\sigma}}|0\rangle &= 0.\end{aligned} \tag{6.161}$$

Using the nonzero expectation value, we can formally write

$$\langle 0|\mathbb{F}^2(x_\mu)|0\rangle = \frac{1}{a^3}\sum_{\mathbf{k},\sigma}\sum_{\bar{\mathbf{k}},\bar{\sigma}}\mathcal{F}(\omega)\mathcal{F}^*(\bar{\omega})\langle 0|a_{\mathbf{k}\sigma}a^\dagger_{\bar{\mathbf{k}}\bar{\sigma}}|0\rangle(e^{\mathbf{k}}_{\perp\sigma}\cdot e^{-\bar{\mathbf{k}}}_{\perp\bar{\sigma}})e^{i(k^\mu-\bar{k}^\mu)x_\mu}. \tag{6.162}$$

The only nonzero value can be derived explicitly by using the commutation relation in Equation 6.57:

$$a_{\mathbf{k}\sigma}a^\dagger_{\bar{\mathbf{k}}\bar{\sigma}} = a^\dagger_{\mathbf{k}\sigma}a_{\bar{\mathbf{k}}\bar{\sigma}} + \delta^3_{\mathbf{k}\bar{\mathbf{k}}}\delta_{\sigma\bar{\sigma}}. \tag{6.163}$$

We then find that

$$\begin{aligned}\langle 0|a_{\mathbf{k}\sigma}a^\dagger_{\bar{\mathbf{k}}\bar{\sigma}}|0\rangle &= \langle 0|a^\dagger_{\mathbf{k}\sigma}a_{\bar{\mathbf{k}}\bar{\sigma}} + \delta^3_{\mathbf{k}\bar{\mathbf{k}}}\delta_{\sigma\bar{\sigma}}|0\rangle\\ &= \langle 0|a^\dagger_{\mathbf{k}\sigma}a_{\bar{\mathbf{k}}\bar{\sigma}}|0\rangle + \langle 0|\delta^3_{\mathbf{k}\bar{\mathbf{k}}}\delta_{\sigma\bar{\sigma}}|0\rangle\\ &= \delta^3_{\mathbf{k}\bar{\mathbf{k}}}\delta_{\sigma\bar{\sigma}}\langle 0|0\rangle\\ &= \delta^3_{\mathbf{k}\bar{\mathbf{k}}}\delta_{\sigma\bar{\sigma}}.\end{aligned} \tag{6.164}$$

Using this last result into Equation 6.162, we have

$$\begin{aligned}\langle 0|\mathbb{F}^2(x_\mu)|0\rangle &= \frac{1}{a^3}\sum_{\mathbf{k},\sigma}\sum_{\bar{\mathbf{k}},\bar{\sigma}}\mathcal{F}(\omega)\,\mathcal{F}^*(\bar{\omega})\,\delta^3_{\mathbf{k}\bar{\mathbf{k}}}\delta_{\sigma\bar{\sigma}}(e^{\mathbf{k}}_{\perp\sigma}\cdot e^{-\bar{\mathbf{k}}}_{\perp\bar{\sigma}})e^{i(k^\mu-\bar{k}^\mu)x_\mu} \\ &= \frac{1}{a^3}\sum_{\mathbf{k},\sigma}\mathcal{F}(\omega)\,\mathcal{F}^*(\omega) \\ &= \frac{1}{a^3}\sum_{\mathbf{k},\sigma}|\mathcal{F}(\omega)|^2 \\ &= \frac{2}{a^3}\sum_{\mathbf{k}}|\mathcal{F}(\omega)|^2. \end{aligned} \tag{6.165}$$

Thus, we obtain the sought-after value for the vacuum fluctuations:

$$\Delta\langle\mathbb{F}\rangle = \sqrt{\frac{2}{a^3}\sum_{\mathbf{k}}|\mathcal{F}(\omega)|^2}. \tag{6.166}$$

If we use an infinite series of modes, the series diverges; this is the purely quantum divergence of the vacuum encountered in QED. Introducing a high cutoff frequency allows one to effectively truncate the series, thus yielding a finite result. The other QED divergences include the vacuum polarization problem and the classical Coulomb divergence of the field energy for a point charge; the latter will be discussed in Chapter 10.

6.10 The Einstein–Podolsky–Rosen Paradox

The Einstein–Podolsky–Rosen, or EPR, paradox is related to the question of locality in quantum mechanics, as described mathematically by Bell's inequalities. An excellent presentation of this problem is given by Mandel and Wolf, as referenced in the bibliography, and we will restrict our discussion to a basic outline of the ideas underlying the EPR paradox.

The basic idea behind the EPR paradox can be described as follows: for initially correlated, or entangled, two-particle states, such as photons produced in a cascade with $\Delta J = 0$, the measurement of a variable on the first particle completely predetermines the result of the measurement of the corresponding variable on the second particle, independent of the space–time distance at which the particles are located at the time of measurement. Quantum mechanically, the entangled state can be represented by

$$|\zeta\rangle = \frac{1}{\sqrt{2}}(|\xi\rangle_1|\psi\rangle_2 - |\psi\rangle_1|\xi\rangle_2), \tag{6.167}$$

where the numbers 1 and 2 refer to each particle, and where all wavefunctions are normalized.

The difficulty with the EPR paradox arises because of the nonlocal character of the entangled state, which allows the observer to know the state of the second particle without a direct measurement or even the time for the measurement performed on the first particle to perturb the second particle.

A classical analog can be constructed by considering an initial correlated state. For example, one could slice a coin along its middle plane, so that the head and tail are separated. Each half of the original coin can be placed in a box, which can then be transported over a great distance, say one to New York and one to Paris. When one experimentalist opens one of the boxes and looks at the side of the coin that is enclosed, she also immediately knows the "state" of the coin on the other side of the Atlantic. The initial correlation remains, allowing for an instantaneous correlation. Note, however, that no information is transported faster than light in the process. Furthermore, we should strongly emphasize the fact that there are fundamental differences between the classical and quantum cases. In particular, in the quantum experiment, the polarization can be measured against an arbitrary reference axis, provided that this axis is contained in a plane perpendicular to the direction of propagation of the photon.

The EPR paradox can be illustrated by considering an experiment with correlated photons produced in a cascade decay. This example is useful because it closely approximates experiments performed by Aspect and his group, and Mandel and Ou, which clearly demonstrated a violation of Bell's inequality, thus ruling out any hidden-variable interpretation of the EPR paradox. In this analysis, we closely follow the presentation of Mandel and Wolf and strongly encourage the reader to consult their book on optical coherence and quantum optics, referenced in the bibliography, for an in-depth discussion of the subject. Two photons polarized orthogonally are considered:

$$|\zeta\rangle = \frac{1}{\sqrt{2}}(|1_{1x}, 0_{1y}, 0_{2x}, 1_{2y}\rangle - |0_{1x}, 1_{1y}, 1_{2x}, 0_{2y}\rangle). \tag{6.168}$$

Here, the $|1_{1x}, 0_{1y}, 0_{2x}, 1_{2y}\rangle$ state corresponds to the first photon polarized along the x-axis, while the second photon is necessarily polarized along the perpendicular axis, the y-axis, while the $|0_{1x}, 1_{1y}, 1_{2x}, 0_{2y}\rangle$ state describes the first photon being polarized along the y-axis, and the second one, necessarily, parallel to the x-axis. The z-axis corresponds to the direction of propagation of photon 1, while photon 2 propagates in the opposite direction. In the entangled state $|\zeta\rangle$, the direction of polarization of an individual photon is unknown, but their polarization states are 100% coupled.

A linear polarizer is inserted along the path of each photon, characterized by the angle $\theta_{1,2}$ with respect to the x-axis, and a detector is positioned after each polarizer; the quantum efficiency of each detector is $\eta_{1,2}$. We now compute the probabilities $P_i(\theta_i)$ of detection of each photon, when the respective polarizers are set at the angles θ_i:

$$P_1(\theta_1) = \eta_1 \langle \zeta | \hat{a}_1^{\dagger} \hat{a}_1 | \zeta \rangle = \frac{\eta_1}{2},$$
$$P_2(\theta_2) = \eta_2 \langle \zeta | \hat{a}_2^{\dagger} \hat{a}_2 | \zeta \rangle = \frac{\eta_2}{2}. \quad (6.169)$$

Here, we have introduced the field dynamical variables,

$$\hat{a}_i = \hat{a}_{ix} \cos\theta_i + \hat{a}_{iy} \sin\theta_i, \quad (6.170)$$

which simply reflect the effect of the polarizers on the photons. The results obtained in Equation 6.169 are readily understood: the probability for each randomly polarized photon to pass through the corresponding polarizer is $\frac{1}{2}$, and the quantum efficiency of the detectors reduces the probability of detection, as reflected in Equation 6.169; these probabilities are independent of the polarizers' settings. Let us now consider the more interesting joint detection probability,

$$\begin{aligned} P_{12}(\theta_1, \theta_2) &= \eta_1 \eta_2 \langle \zeta | \hat{a}_1^{\dagger} \hat{a}_2^{\dagger} \hat{a}_2 \hat{a}_1 | \zeta \rangle \\ &= \frac{1}{2} \eta_1 \eta_2 (\sin^2\theta_1 \cos^2\theta_2 + \cos^2\theta_1 \sin^2\theta_2 - 2\sin\theta_1 \cos\theta_2 \cos\theta_1 \sin\theta_2) \\ &= \frac{1}{2} \eta_1 \eta_2 \sin^2(\theta_1 - \theta_2), \end{aligned} \quad (6.171)$$

which clearly introduces a correlation between the polarizers.

Using random variable analysis, the conditional detection probability of the second photon can be expressed in terms of the probability detection of the first photon:

$$\frac{P_{12}(\theta_1, \theta_2)}{P_1(\theta_1)} = \eta_2 \sin^2(\theta_1 - \theta_2). \quad (6.172)$$

If the quantum efficiency of the detector for photon 2 is close to 100% and the polarizers are set orthogonally, so that $\theta_1 - \theta_2 = \pm\pi/2$, Equation 6.172 shows that the conditional detection probability of the second photon approaches 100%: the photons are clearly polarized orthogonally. Furthermore, the polarization state of the second photon can be known by determining that of photon 1, without a measurement on photon 2, and instantaneously, independent of the separation between the two photons at the time of the measurement on the first photon. However, as shown by Mandel and Wolf, causality is preserved, as the polarization axis of the first photon is not set by the direction of polarizer 1; it merely serves as a reference axis to measure a random variable.

In closing, we outline the derivation of Bell's inequality, where two observables, A and B, parameterized by the variables α and β, are considered. Moreover, measurements of A and B can only yield two possible values, say 0 or 1. For example, in the case of polarizers the photons can either be transmitted or absorbed, and the parameter is the angle of the polarizer, θ.

Bell considers the average correlation between the observables:

$$C(\alpha,\beta) = \langle A(\alpha)B(\beta)\rangle. \tag{6.173}$$

The key idea behind Bell's derivation is to test the validity of so-called "hidden variable" theories; therefore, the correlation in Equation 6.173 is explicitly expressed as

$$C(\alpha,\beta) = \int A(\alpha,\eta)B(\alpha,\eta)\rho(\eta)\,d\eta, \tag{6.174}$$

where η represents the hidden variable, while $\rho(\eta)$ is its normalized probability density:

$$\int \rho(\eta)\,d\eta = 1. \tag{6.175}$$

Locality is implicit in Equation 6.174, in the sense that A does not depend on β, while B does not depend on α.

We now examine the quantities $|C(\alpha, \beta) - C(\alpha, \beta')|$ and $|C(\alpha', \beta) + C(\alpha', \beta')|$. We first have

$$\begin{aligned} |C(\alpha, \beta) - C(\alpha, \beta')| &\leq \int |A(\alpha, \eta)[B(\beta, \eta) - B(\beta', \eta)]|\rho(\eta)\,d\eta, \\ &\leq \int |B(\beta, \eta) - B(\beta',\eta)|\rho(\eta)\,d\eta, \end{aligned} \tag{6.176}$$

since, by definition, $|A(\alpha, \eta)| = 1$. For the same reason, namely $|A(\alpha',\eta)| = 1$, we also have

$$\begin{aligned} |C(\alpha', \beta) + C(\alpha', \beta')| &\leq \int |A(\alpha', \eta)[B(\beta, \eta) + B(\beta', \eta)]|\rho(\eta)d\eta, \\ &\leq \int |B(\beta, \eta) + B(\beta', \eta)|\rho(\eta)\,d\eta. \end{aligned} \tag{6.177}$$

Adding Equations 6.176 and 6.177 together, we can thus write

$$\begin{aligned} &|C(\alpha, \beta) - C(\alpha, \beta')| + |C(\alpha', \beta) + C(\alpha', \beta')| \\ &\quad \leq \int [|B(\beta, \eta) - B(\beta', \eta)| + |B(\beta, \eta) + B(\beta', \eta)|]\rho(\eta)\,d\eta. \end{aligned} \tag{6.178}$$

Furthermore, since $|B(\beta, \eta)| = |B(\beta',\eta)| = 1$, we have

$$|B(\beta, \eta) - B(\beta', \eta)| + |B(\beta, \eta) + B(\beta', \eta)| = 2. \tag{6.179}$$

Using the result given in Equation 6.179, together with the normalization of the hidden variable probability density, described in Equation 6.175, we obtain Bell's inequality:

$$|C(\alpha,\beta) - C(\alpha,\beta')| + |C(\alpha',\beta) + C(\alpha',\beta')| \le 2. \tag{6.180}$$

Note that the hidden variable has now disappeared, by virtue of the integration. Bell's inequality provides a test of the correlation between dichotomic observables, such as those discussed in the case of correlated photons being analyzed by linear polarizers. Using the transmission probability through a polarizer, which can be obtained by setting the quantum efficiencies equal to unity ($\eta_1 = \eta_2 = 1$), we have

$$\begin{aligned} C(\theta_1,\theta_2) &= \frac{1}{2}[\sin^2(\theta_1-\theta_2) + \sin^2(\theta_1-\theta_2) - \cos^2(\theta_1-\theta_2) - \cos^2(\theta_1-\theta_2)] \\ &= \sin^2(\theta_1-\theta_2) - \cos^2(\theta_1-\theta_2) \\ &= -\cos[2(\theta_1-\theta_2)]. \end{aligned} \tag{6.181}$$

It is clear that for a careful choice of polarizing angles, we can set up a violation of Bell's inequality; in other words, we can find a set of angles, θ_1, θ_2, θ_1', θ_2', such that

$$\begin{aligned} &|C(\theta_1,\theta_2) - C(\theta_1,\theta_2')| + |C(\theta_1',\theta_2) + C(\theta_1',\theta_2')| \\ &\quad = |-\cos[2(\theta_1-\theta_2)] + \cos[2(\theta_1-\theta_2')]| + |-\cos[2(\theta_1'-\theta_2)] - \cos[2(\theta_1'-\theta_2')]| \\ &\quad = |-\cos[2(\theta_1-\theta_2)] + \cos[2(\theta_1-\theta_2')]| + |\cos[2(\theta_1'-\theta_2)] + \cos[2(\theta_1'-\theta_2')]| \\ &\quad > 2. \end{aligned} \tag{6.182}$$

For example, the angles $\theta_1 = 0$, $\theta_2 = \pi/8$, $\theta_2' = 3\pi/8$, and $\theta_1' = \pi/4$, yield $2\sqrt{2}$, in clear violation of Bell's inequality. Such experiments have been performed, ruling out the possibility of local, hidden-variable theories for quantum mechanics.

6.11 Squeezed States

This interesting example of quantum effects in nonlinear optics is closely related to other subjects, including quantum nondemolition (QND) measurements, optical phase conjugation and degenerate four-wave mixing, as well as parametric down-conversion and the production of entangled quantum states. In the same general field of modern physics, we find topics such as cavity QED, the slowing down of light to extremely low speeds and its storage as a coherent spin state in atoms, as well as Bose–Einstein condensation. A number of recent publications and review articles on these very interesting developments are referenced in the bibliography.

The basic idea behind squeezed states is that conjugate variables for the vacuum state satisfy Heisenberg's uncertainty principle,

$$\Delta p \Delta q \geq \hbar, \tag{6.183}$$

in a very specific way. For properly normalized variables, we have $\Delta p = \Delta q = \sqrt{\hbar}$. Within this context, a squeezed state will have one of its conjugate variables below the vacuum level, while the other variable will be above the vacuum level, to properly satisfy the uncertainty principle. In phase space, this corresponds to a circle of surface $\hbar$ for the vacuum state, and to an ellipse, with the same surface, for the squeezed state.

More precisely, if we consider the normalized operators

$$\begin{aligned} \hat{q} &= \hat{a}^{\dagger} + \hat{a}, \\ \hat{p} &= i(\hat{a}^{\dagger} - \hat{a}), \end{aligned} \tag{6.184}$$

which are defined in terms of the creation and annihilation operators and correspond to generalized coordinates and momenta, we find that their commutator is

$$[\hat{q}, \hat{p}] = 2i, \tag{6.185}$$

while the corresponding Heisenberg uncertainty relation takes the form

$$\sqrt{\langle \Delta\hat{q}^2 \rangle \langle \Delta\hat{p}^2 \rangle} \geq 1. \tag{6.186}$$

With these definitions, a squeezed state can be constructed mathematically by introducing a phase angle, θ, and introducing the new operators

$$\begin{aligned} \widehat{q} &= \hat{a}^{\dagger} e^{i\theta} + \hat{a} e^{-i\theta}, \\ \widehat{p} &= i(\hat{a}^{\dagger} e^{i\theta} - \hat{a} e^{-i\theta}). \end{aligned} \tag{6.187}$$

It is then easily seen that

$$\begin{aligned} \Delta\widehat{q} &= \Delta\hat{q}\cos\theta + \Delta\hat{p}\sin\theta, \\ \Delta\widehat{p} &= -\Delta\hat{q}\sin\theta + \Delta\hat{p}\cos\theta, \end{aligned} \tag{6.188}$$

which also lead to Heisenberg's uncertainty relation,

$$\sqrt{\langle \Delta \hat{q}^2 \rangle \langle \Delta \hat{p}^2 \rangle} \geq 1; \tag{6.189}$$

however, it is clear that for some values of θ, we can have $\langle \Delta \hat{q}^2 \rangle < 1$, $\langle \Delta \hat{p}^2 \rangle > 1$, or $\langle \Delta \hat{q}^2 \rangle > 1$, $\langle \Delta \hat{p}^2 \rangle < 1$, thus constituting a squeezed state.

Degenerate four-wave mixing gives rise to squeezed states, as do other nonlinear interactions. This particular interaction results from the mixing of two input or pump waves in a $\chi^{(3)}$ medium, producing two output signals: a signal and an idler wave. The pump waves can typically be treated classically, while the output signals exhibit quantum mechanical features, including squeezing and optical phase conjugation.

Closely related and of considerable interest is the concept of quantum nondemolition (QND) measurements, where a particular variable, $\hat{\Theta}$, is created experimentally, which obeys the commutation relation

$$[\hat{\Theta}(t_1), \hat{\Theta}(t_2)] = 0, \quad \forall t_1, t_2. \tag{6.190}$$

This means that measurements of this particular variable at different times will yield the same result: the variable is not influenced by the measurement process. A good example of an experimental situation producing a QND variable is the Kerr effect, which also involves a $\chi^{(3)}$ nonlinearity.

For detailed discussions of these concepts, we refer the reader to Mandel and Wolf, as well as the articles listed in the bibliography.

6.12 Casimir Effect

The quantum vacuum fluctuations described in Section 6.9 give rise to an interesting phenomenon, the Casimir effect. In the presence of boundary conditions, the mode structure of the vacuum excitations is modified, as a discrete spectrum emerges, with a cutoff frequency, instead of the continuum of free space; in turn, this produces a differential radiation pressure, which is manifested as a force on the boundary surface.

The simple case of two parallel conducting plates is considered here, for the sake of illustration. If the surface of the plates is much larger than their separation ($\sqrt{S} \gg \Delta z$), we can consider that the minimum axial wavenumber will be given by

$$k\Delta z = \pi. \tag{6.191}$$

We can then compute the vacuum energy in the cavity:

$$\begin{aligned} W &= \sum_{|\mathbf{k}| \geq \pi/\Delta z, \sigma} \frac{1}{2}\hbar\omega \\ &\simeq \Delta z S \int_{\pi/\Delta z}^{k*} (\hbar c k) k^2 dk \\ &= \hbar c \Delta z S \left[\frac{k^4}{4}\right]_{\pi/\Delta z}^{k*} \\ &= \hbar c \Delta z S \frac{k^{*4}}{4} - \hbar c S \frac{\pi^4}{4} \frac{1}{\Delta z^3} \\ &= W^* - W_0. \end{aligned} \tag{6.192}$$

Here, we have approximated the sum by an integral and introduced a high wavenumber cutoff, k^*, to avoid divergences. The Casimir pressure, P, is given by deriving the work of the force on the plates required to balance the variation of the energy between the plates. We have

$$PSd\Delta z + dW_0 = Fd\Delta z + dW_0 = 0, \tag{6.193}$$

which yields

$$P = -\frac{1}{S}\frac{dW_0}{d\Delta z} = \frac{\hbar c \pi^4}{\Delta z^4}. \tag{6.194}$$

Although the numerical factor is wrong, the scaling of the force with the plate separation is correct and has been measured experimentally. We also note that, depending on the type of boundary condition, for example conductor or dielectric, the pressure can be positive or negative. Furthermore, the exact scaling of the Casimir force is related to the dimensionality of space–time, as probed by the quantum vacuum modes. Finally, it has been speculated that this type of effect can give rise to so-called "false vacuum" states, with negative energy densities giving rise to a cosmological constant. It has also been proposed by Thorne and co-authors that stable wormholes and time machines could be built from such false vacua.

6.13 Reflection of Plane Waves in Rindler Space

Most of the text and derivations in this section were produced by J. R. Van Meter.

6.13.1 Background

In recent years, much attention has been given to the interaction of uniformly accelerating systems with quantum fields, particularly with regards to the thermal Fulling–Davies–Unruh radiation. In contrast, relatively little attention has been given to the interaction of uniformly accelerating systems with classical fields. However, the latter domain seems deserving of study for several reasons. First, this subject represents physics fundamental to both classical electrodynamics and general relativity. For example, it is instructive to study a uniformly accelerating charged particle in a classical context to understand how a radiation field in inertial coordinates can appear as a static field in accelerated coordinates, as well as to explore the approximate behavior of the Coulomb field in Schwarzschild space–time. More generally, because of the mathematical similarity between Rindler and Kruskal coordinates, any result obtained for a uniformly accelerated system may be extendable, at least qualitatively, to a corresponding system in the vicinity of a black hole horizon (as already demonstrated by the deep parallels between Fulling–Davies–Unruh radiation and Hawking radiation).

Another motivation for studying the interaction of uniformly accelerated systems with classical fields is that such analyses might shed some light on corresponding problems in quantum field theory. Boyer's program of approximating quantum electrodynamics with the semiclassical model of stochastic electrodynamics (SED) is noteworthy in this context. In the methodology of SED, the quantum electrodynamical vacuum is approximated by an infinite sum over momenta of plane waves, each with a random phase and an infinitesimal amplitude calculated so as to give a total energy per plane wave of $\frac{1}{2}\hbar\omega$. This model has proven very interesting, as one can match quantum electrodynamical results when calculating the Casimir effect for various boundary configurations. It appears that this model may also be used to derive the thermal effects on a system accelerating uniformly through vacuum, in agreement with quantum field theory.

Of particular interest here is the question of whether Fulling–Davies–Unruh radiation can be backscattered into an inertial laboratory frame, and the possibility of addressing this issue within the framework of semiclassical vacuum fluctuations in Rindler space–time. Various proposals have been put forth for laboratory measurements of backscattered Fulling–Davies–Unruh radiation, including that of Tajima and Chen utilizing an ultrahigh intensity laser to strongly accelerate electrons. Despite some controversy and slight confusion in the literature, the emerging consensus amongst quantum field theorists seems to be that a uniformly accelerating system will not measurably reradiate Fulling–Davies–Unruh radiation into an inertial frame. However, these studies only considered scalar vacuum fields; whether this null radiation result holds for the case of the electromagnetic tensor field has yet to be demonstrated theoretically or experimentally. Whether a uniformly accelerating system will reradiate Fulling–Davies–Unruh radiation into the inertial lab frame thus remains an open question of modern physics. The problem explored in this section might prove germane to the issue.

The present discussion considers the interaction of a uniformly accelerating, perfectly conducting plane mirror with a plane wave at normal incidence. In this regard, the present study is self-contained and represents an original contribution to fundamental classical electrodynamics, particularly by providing physical insight into the relationship between Rindler and Lorentz transformations. This work is especially motivated by its potential relevance to the case of a uniformly accelerating mirror interacting with a quantum field in the vacuum state.

The pertinence of this analysis to the problem of an accelerating mirror interacting with the quantum vacuum may be understood within the stochastic electrodynamical framework as follows. In this model, each virtual photon plane wave incident on the mirror will give rise to a reflected wave that might or might not interfere significantly with the original incident wave. However, each pair of incident/reflected waves will not interfere significantly with any other wave, because of the relative randomization of phases that characterizes the stochastic electrodynamical approach. Thus, in computing the total spectrum, the waves will add incoherently, with the possible exception of each incident wave with its corresponding reflected wave. For the purpose of predicting the qualitative character of the spectrum, it should therefore suffice to consider only an individual incident wave and its reflected wave. The simplest case of normal incidence is the most natural starting point for such an inquiry.

The incident and reflected fields are first transformed to Rindler coordinates and the boundary condition imposed by the mirror, now fixed at a stationary position in Rindler space, is found to determine the reflected wave function. The reflected wave is then expressed in Minkowski coordinates, where its physical meaning is more readily interpreted. To further explicate the physics involved, an alternative means of solving for the reflected wave is presented, which utilizes the Lorentz transform as well as a simple strategy for handling retardation that exploits the unique geometric properties of this problem. Both the case where the mirror accelerates uniformly for all time and the case where the mirror is initially at rest and starts accelerating at $t = 0$ are considered in this section. Finally, some implications of these results are discussed.

6.13.2 Derivation of the Reflected Wave Using the Rindler Transform

The problem outlined in the introduction can be summarized more precisely as follows. A mirror moves with uniform proper acceleration such that

$$a_\nu a^\nu = \frac{du_\nu}{d\tau}\frac{du^\nu}{d\tau} = \left(\frac{d^2\mathbf{x}}{d\tau^2}\right)^2 - \left(\frac{d^2t}{d\tau^2}\right)^2 \equiv a^2, \tag{6.195}$$

where a is a constant, and where τ is the proper time along the mirror's world line $x_\nu(\tau)$, $u_\nu = dx_\nu/d\tau$ is the mirror four-velocity, and a_ν is its four-acceleration; note that units are normalized so that the speed of light is equal to 1.

We are considering a one-dimensional problem, since the incident and reflected electromagnetic radiation are plane waves at normal incidence; thus Equation 6.195 reduces to

$$a_\nu a^\nu = \left(\frac{\partial^2 z}{\partial \tau^2}\right)^2 - \left(\frac{\partial^2 t}{\partial \tau^2}\right)^2 \equiv a^2. \tag{6.196}$$

To simplify later results, we set $z = a^{-1}$ at $t = 0$. Note that for $t < 0$, $dz/dt < 0$, while for $t > 0$, $dz/dt > 0$. The more realistic case where $dz/dt = 0$ for $t < 0$ will be explored in Section 6.13.3.

A plane wave with wave vector $\mathbf{k} = -k\hat{z}$ is incident on the mirror. Given the geometry of this problem, the electromagnetic field tensor reduces to

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & 0 & 0 \\ -E_x & 0 & 0 & -B_y \\ 0 & 0 & 0 & 0 \\ 0 & B_y & 0 & 0 \end{bmatrix}. \tag{6.197}$$

The incident wave is then given by

$$E_x^I = -B_y^I = E_0 \cos(-kz - \omega t) = E_0 \cos[k(z+t)], \tag{6.198}$$

while the reflected wave can be assumed to be of the form

$$E_x^R = B_y^R = -E_0 f(z-t). \tag{6.199}$$

It is easily seen that Equations 6.198 and 6.199 satisfy Maxwell's equations.

We now consider the Rindler transform, which allows us to study the incident and reflected waves in an accelerated frame where the mirror is at rest at all times. Rindler coordinates are related to Minkowski coordinates by

$$\bar{z} = \sqrt{z^2 - t^2}, \quad \bar{t} = \frac{1}{2a}\ln\left(\frac{z+t}{z-t}\right), \tag{6.200}$$

and

$$z = \bar{z}\cosh(a\bar{t}), \quad t = \bar{z}\sinh(a\bar{t}), \tag{6.201}$$

where the coordinate transform has been scaled according to the mirror's acceleration, for convenience.

The Minkowski metric may be transformed to the Rindler metric:

$$ds^2 = -a^2\bar{z}^2 d\bar{t}^2 + dx^2 + dy^2 + d\bar{z}^2. \tag{6.202}$$

The electromagnetic field tensor may be transformed by the well-known formula

$$\bar{F}^{\mu\nu} = \frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}F^{\alpha\beta}, \tag{6.203}$$

which yields

$$\begin{aligned}
\bar{F}^{01} &= -\bar{F}^{10} = \frac{\partial\bar{t}}{\partial t}F^{01} + \frac{\partial\bar{t}}{\partial z}F^{31} = \frac{\cosh(a\bar{t})}{a\bar{z}}E_x - \frac{\sinh(a\bar{t})}{a\bar{z}}B_y, \\
\bar{F}^{31} &= -\bar{F}^{13} = \frac{\partial\bar{z}}{\partial t}F^{01} + \frac{\partial\bar{z}}{\partial z}F^{31} = -\sinh(a\bar{t})E_x + \cosh(a\bar{t})B_y, \\
\bar{F}^{\mu\nu} &= 0 \ \ \text{otherwise.}
\end{aligned} \tag{6.204}$$

With this, the incident and reflected phase variables become

$$z + t = -\bar{z}\cosh(a\bar{t}) - \bar{z}\sinh(a\bar{t}) = \bar{z}\exp(a\bar{t}), \tag{6.205}$$

and

$$z - t = \bar{z}\cosh(a\bar{t}) - \bar{z}\sinh(a\bar{t}) = \bar{z}\exp(-a\bar{t}). \tag{6.206}$$

We thus obtain

$$\begin{aligned}
\bar{E}_x^I &= E_0\frac{\exp(a\bar{t})}{a\bar{z}}\cos[k\bar{z}\exp(a\bar{t})], \\
\bar{B}_y^I &= -E_0\exp(a\bar{t})\cos[k\bar{z}\exp(a\bar{t})],
\end{aligned} \tag{6.207}$$

and

$$\begin{aligned}
\bar{E}_x^R &= -E_0\frac{\exp(-a\bar{t})}{a\bar{z}}f[\bar{z}\exp(-a\bar{t})], \\
\bar{B}_y^R &= -E_0\exp(-a\bar{t})f[\bar{z}\exp(-a\bar{t})].
\end{aligned} \tag{6.208}$$

It is easy to check whether these expressions satisfy the generally covariant form of Maxwell's equations in vacuum, $\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0$, and $F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu} = 0$, which, in the one-dimensional geometry of this problem, reduce to

$$\frac{\partial}{\partial\bar{t}}(a\bar{z}\bar{E}_x) + \frac{\partial}{\partial\bar{z}}(a\bar{z}\bar{B}_y) = 0, \tag{6.209}$$

and

$$\frac{\partial}{\partial \bar{t}}\bar{B}_y + \frac{\partial}{\partial \bar{z}}(a^2\bar{z}^2\bar{E}_x) = 0. \tag{6.210}$$

At this point, we note that the boundary condition for a perfect conductor mandates that there be no transverse electromagnetic forces on the electrons within the mirror. Mathematically, this condition is expressed as

$$e\bar{F}^{1\nu}\bar{u}_\nu\Big|_{\bar{z}=\frac{1}{a}} = e\bar{F}^{2\nu}\bar{u}_\nu\Big|_{\bar{z}=\frac{1}{a}} = 0, \tag{6.211}$$

where $\bar{u}_\nu = d\bar{x}_\nu/d\tau$.

Since the field is transverse and $\bar{u}_3 = 0$, we have

$$\bar{E}_x\Big|_{\bar{z}=\frac{1}{a}} = (\bar{E}_x^I + \bar{E}_x^R)\Big|_{\bar{z}=\frac{1}{a}} = 0. \tag{6.212}$$

In order to solve for the unknown function f in the expression for the reflected wave, the incident and reflected electric fields in Equations 6.207 and 6.208 can now be used in Equation 6.212 to yield

$$E_0\exp(a\bar{t})\cos\left[\frac{k}{a}\exp(a\bar{t})\right] - E_0\exp(-a\bar{t})f[\bar{z}\exp(-a\bar{t})]\Big|_{\bar{z}=\frac{1}{a}} = 0. \tag{6.213}$$

A little algebra reveals that the only value for f, which satisfies Equation 6.213 while maintaining its space–time dependence exclusively on $\bar{z}\exp(-a\bar{t})$, in order to satisfy Maxwell's equations, is

$$f[\bar{z}\exp(-a\bar{t})] = \frac{\exp(2a\bar{t})}{a^2\bar{z}^2}\cos\left[\frac{k\exp(a\bar{t})}{a^2\bar{z}}\right]. \tag{6.214}$$

The reflected wave in Rindler coordinates thus becomes

$$\bar{E}_x^R = -E_0\frac{\exp(a\bar{t})}{a^3\bar{z}^3}\cos\left[\frac{k\exp(a\bar{t})}{a^2\bar{z}}\right], \tag{6.215}$$

and

$$\bar{B}_y^R = -E_0\frac{\exp(a\bar{t})}{a^2\bar{z}^2}\cos\left[\frac{k\exp(a\bar{t})}{a^2\bar{z}}\right]. \tag{6.216}$$

Using Equation 6.206, the function f can be expressed in terms of Minkowski coordinates as

$$f(z-t) = \frac{\cos\left[\frac{k}{a^2(z-t)}\right]}{a^2(z-t)^2}, \tag{6.217}$$

and the reflected wave becomes

$$E_x^R = B_x^R = -E_0\frac{\cos\left[\frac{k}{a^2(z-t)}\right]}{a^2(z-t)^2}. \tag{6.218}$$

The physical reason for the unusual dependence on $z - t$ will be made clear in the next section. Here, we only point out that the apparent singularity in the field at $z = t$ does not pose any difficulty. For any finite time t, the position of the mirror in our coordinates is greater than t: $z_m(t) > t$; thus the point $z = t$ always lies behind the mirror, opposite to the side on which the plane wave is incident, and therefore outside the region for which Equation 6.218 is valid.

6.13.3 Derivation of the Reflected Wave Using the Lorentz Transform

We first consider a plane wave at normal incidence to a mirror with constant velocity. In order to solve for the reflected wave, the problem is treated most easily in the frame of the mirror, which requires a Lorentz transform of the original expression for the incident wave:

$$E_x'^I = \gamma(E_x^I - \beta B_y^I) = \gamma(1-\beta)E_x^I, \tag{6.219}$$

and

$$B_y'^I = \gamma(B_y^I - \beta E_x^I) = \gamma(1-\beta)B_y^I, \tag{6.220}$$

where β is the relative velocity between the instantaneous rest frame of the mirror and the reference frame.

Since we have

$$z + t = \gamma(z' + \beta t') + \gamma(t' + \beta z') = \gamma(1+\beta)(z' + t'), \tag{6.221}$$

the incident wave can be expressed as

$$E_x'^I(z', t') = -B_y'^I(z', t') = \gamma(1+\beta)E_0\cos[\gamma(1+\beta)k(z'+t')]. \tag{6.222}$$

The boundary condition now reads

$$\mathbf{n} \times \mathbf{E}'|_{z'=z'_m} = \hat{y}(E'^I_x + E'^R_x)\big|_{z'=0} = 0, \tag{6.223}$$

which implies,

$$\mathbf{E}'^R\big|_{z'=0} = -\hat{x}\gamma(1+\beta)E_0\cos[\gamma(1+\beta)k(-t')]. \tag{6.224}$$

To satisfy both the boundary condition and Maxwell's equations, the reflected wave can only take the form

$$E'^R_x(z',t') = B'^R_y(z',t') = -\gamma(1+\beta)E_0\cos[\gamma(1+\beta)k(z'-t')]. \tag{6.225}$$

Lorentz transforming the reflected wave back to the original lab frame and noting that

$$z'-t' = \gamma(z-\beta t) - \gamma(t-\beta z) = \gamma(1+\beta)(z-t), \tag{6.226}$$

the reflected wave is found to be

$$E^R_x(z,t) = B^R_y(z,t) = -\gamma^2(1+\beta)^2E_0\cos[\gamma^2(1+\beta)^2k(z-t)]. \tag{6.227}$$

To extend this result to the case of an accelerating mirror, we observe that a ray of light reflected from an accelerating mirror at some time t_r and position z_r, where it has velocity β, is indistinguishable from a ray of light reflected from an identical mirror at the same time t_r and the same position z_r, but with a constant velocity β_0 which happens to equal β at that instant.

The retarded position z_r and retarded time t_r can be expressed in terms of the retarded proper time τ_r of the mirror as follows:

$$z_r = \frac{1}{a}\cosh(a\tau_r), \quad t_r = \frac{1}{a}\sinh(a\tau_r). \tag{6.228}$$

The retarded Lorentz boost parameters γ and β thus satisfy the following relations:

$$\gamma\beta = \frac{dz_r}{d\tau_r} = \sinh(a\tau_r), \quad \gamma = \frac{dt_r}{d\tau_r} = \cosh(a\tau_r). \tag{6.229}$$

Hence, using Equations 6.228 and 6.229, and recalling the identity for hyperbolic functions, $\cosh^2 s - \sinh^2 s = 1$, we find

$$(\gamma + \gamma\beta)^2 = \frac{1}{a^2(z_r - t_r)^2}. \tag{6.230}$$

Invoking the light-cone condition, $z_r - t_r = z - t$, the following identity is finally obtained:

$$(\gamma + \gamma\beta)^2 = \frac{1}{a^2(z-t)^2}. \tag{6.231}$$

The reflected wave is therefore described by

$$E_x^R = B_y^R = -E_0 \frac{\cos\left[\frac{k}{a^2(z-t)}\right]}{a^2(z-t)^2}, \tag{6.232}$$

which confirms the result obtained by the Rindler method.

With this approach, several curious features of the reflected wave are understood easily in terms of the Doppler shift. For example, the amplitude of the reflected wave goes to zero as z goes to infinity because of an infinite redshift. At larger z, the observed reflected wave originates from a point farther back in the past on the mirror's world line, when the mirror had larger acceleration away from the observer. The resulting Doppler redshift of the reflected wave therefore increases with z. Another interesting effect to note is that as time increases, the velocity of the mirror asymptotically approaches the speed of light and, correspondingly, its position asymptotically approaches the reflected wave singularity at $t = z$. Thus, the amplitude of the field near the mirror increases with time, which physically is due, of course, to Doppler blue-shifting.

By the reasoning above, the more realistic case in which the mirror is at rest until uniform proper acceleration is initiated at some finite time can be examined readily. Following the previous light-ray argument and considering the retarded quantities, it is clear that if the mirror is at rest for $t < 0$ and begins to accelerate uniformly at $t = 0$, the reflected wave must be

$$E_x^R = B_y^R = \begin{cases} -E_0 \dfrac{\cos\left[\frac{k}{a^2(z-t)}\right]}{a^2(z-t)^2}, & z - t \le \dfrac{1}{a}, \\ -E_0 \cos[k(z-t)], & z - t \ge \dfrac{1}{a}. \end{cases} \tag{6.233}$$

As z increases, the Doppler redshift will decrease the amplitude and frequency of the reflected wave only until $z = t + 1/a$; beyond this point the reflected wave appears as a monochromatic plane wave because its retarded "source" is now stationary.

In conclusion, the reflected wave from a uniformly accelerating mirror has been derived using the Rindler transform and, alternatively, using the Lorentz transform. The physics of the result obtained by the Rindler method have been elucidated by the Lorentz transform approach, and the expected Doppler effects have been plainly demonstrated. Further, both the case where the mirror is always accelerating and the case where the mirror begins acceleration at some finite time have been examined.

We now attempt to interpret these results within the context of SED, wherein the incident wave is taken to represent a virtual photon, so that it has an energy density equal to $\frac{1}{2}\hbar\,\omega d^3k$. Admittedly, in summing over the infinite momenta of the vacuum, it is not completely clear how to compare meaningfully the total infinite spectrum obtained from the incident and reflected fields with the infinite spectrum of an unbounded vacuum. However, for the purposes of making a qualitative prediction, we note that the amplitude of the reflected field becomes arbitrarily large for small $z - t$. Thus, it seems reasonable to assert that, within the framework of this model, a detector stationed at some fixed position z sufficiently larger than $1/a$ will detect a pulse of radiation that is significantly larger in amplitude than the vacuum noise as soon as the mirror approaches sufficiently close. This semiclassical result seems to be in conflict with the quantum treatment of the same problem; this might indicate that the stochastic electrodynamical model breaks down in this situation. However, the full SED and QED calculations must be performed before definitive statements can be made in this regard; perhaps relevant experiments will also be performed in the not-too-distant future.

6.13.4 Mathematical Appendix

It can be shown that the correct plane wave solution is recovered for the reflected wave in the zero-acceleration limit. Taking this limit is not trivial, however, because in the previous expressions it was assumed that the mirror is located at $z = 1/a$ when $t = 0$. To obtain meaningful results in the zero-acceleration limit, it is therefore necessary to shift the z coordinate:

$$z' = z - \frac{1}{a}. \tag{6.234}$$

For this purpose it will simplify matters considerably to use complex fields, such that

$$E_x^I = \mathrm{Re}(\tilde{E}_x^I) = \mathrm{Re}\{E_0 \exp[-ik(z+t)]\}, \tag{6.235}$$

and

$$E_x^R = \mathrm{Re}(\tilde{E}_x^R) = \mathrm{Re}\left\{-E_0 \frac{\exp\left[\frac{-ik}{a^2(z-t)}\right]}{a^2(z-t)^2}\right\}. \tag{6.236}$$

Expressing the incident wave in the new coordinate system,

$$\tilde{E}_x^I = -\tilde{B}_y^I = E_0 \exp\left\{-ik\left[\left(z' + \frac{1}{a}\right) + t\right]\right\} = E_0 \exp\left(-i\frac{k}{a}\right)\exp[-ik(z'+t)], \tag{6.237}$$

the zero-acceleration limit can be taken to yield

$$\lim_{a\to 0}\tilde{E}_x^I = \tilde{E}_0'\exp[-ik(z'+t)], \tag{6.238}$$

where $\tilde{E}_0' \equiv \lim_{a\to 0} E_0 \exp(-i\frac{k}{a})$.

Now expressing the reflected wave in the new coordinate system

$$\tilde{E}_x^R = \tilde{B}_y^R = -E_0 \frac{\exp\left\{\frac{-ik}{a[1+a(z'-t)]}\right\}}{[1+a(z'-t)]^2}, \tag{6.239}$$

the zero-acceleration limit for the reflected wave can be taken as follows:

$$\begin{aligned}
\lim_{a\to 0}\tilde{E}_x^R &= -\lim_{a\to 0} E_0 \exp\left\{\frac{-ik}{a[1+a(z'-t)]}\right\} \\
&= -\lim_{a\to 0} E_0 \exp\left\{-i\frac{k}{a}[1-a(z'-t)]\right\} \\
&= -\lim_{a\to 0} E_0 \exp\left(-i\frac{k}{a}\right)\exp[ik(z'-t)] \\
&= -\tilde{E}_0'\exp[ik(z'-t)].
\end{aligned} \tag{6.240}$$

This result is exactly the reflected wave corresponding to the incident plane wave in Equation 6.239 for a stationary mirror at $z' = 0$.

6.14 References for Chapter 6

Note: the numbers listed below refer to the main bibliography and reference sections at the end of this book.

1, 4, 17, 32, 35, 36, 55, 56, 69, 73, 83, 132, 148, 151, 157, 166, 181, 214, 215, 219, 247, 263, 264, 265, 266, 267, 269, 271, 310, 315, 318, 319, 320, 321, 322, 328, 343, 344, 356, 357, 375, 389, 390, 406, 407, 408, 415, 514, 515, 524, 532, 533, 534, 535, 551, 552, 553, 554, 555, 556, 557, 586, 589, 590, 591, 594, 623, 624, 633, 634, 639, 642, 643, 656, 664, 702, 703, 707, 713, 760, 761, 853, 854, 855, 856, 857, 858, 859, 860, 862, 863, 864, 866, 869, 873, 876, 877, 905, 906, 907.