

# Notes on “A Greater ElectroMagnetism”

M.B. van der Mark<sup>(a)</sup> and J.G. Williamson<sup>(b)</sup>

<sup>(a)</sup>Philips Research Laboratories, HTC 34, WB-21,  
Prof. Holstlaan 4, 5656 AA Eindhoven, The Netherlands

<sup>(b)</sup>Glasgow University, Department of Electronics & Electrical Engineering,  
Glasgow G12 8QQ, Scotland

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## Appendix XV: Multiplication of multivectors using matrices

The multiplication of two general multivectors involves 256 components, and together with its non-commuting nature, this makes it useful to be implemented in a computer program as a matrix multiplication.

$$\begin{aligned}\Psi\Phi &= (M(\Phi)\vec{\Psi}) \cdot \vec{e} \\ &= (M(\Phi)_{XY}\Psi_Y) \cdot e_Y\end{aligned}\quad (1)$$

$$\Psi = \Psi_Y \cdot e_Y \quad Y \in \{, \nu, \mu\nu, \kappa\mu\nu, 0123\} \quad (2)$$

If  $\Psi$  is of the form given by Eq. (??), so that  $s_0 = \phi$ ,  $v_\nu = \phi_\nu$ ,  $b_i = \phi_{i0}$ , etc., the matrix  $M$  looks like this:

$$\left( \begin{array}{cccccccccccccccccc} \phi & \phi_0 & -\phi_1 & -\phi_2 & -\phi_3 & \phi_{10} & \phi_{20} & \phi_{30} & -\phi_{23} & -\phi_{31} & -\phi_{12} & -\phi_{023} & -\phi_{031} & -\phi_{012} & \phi_{123} & -\phi_{0123} \\ \phi_0 & \phi & -\phi_{10} & -\phi_{20} & -\phi_{30} & \phi_1 & \phi_2 & \phi_3 & -\phi_{023} & -\phi_{031} & -\phi_{012} & -\phi_{23} & -\phi_{31} & -\phi_{12} & -\phi_{0123} & \phi_{123} \\ \phi_1 & -\phi_{10} & \phi & \phi_{12} & -\phi_{31} & \phi_0 & \phi_{012} & -\phi_{031} & -\phi_{123} & \phi_3 & -\phi_2 & -\phi_{0123} & -\phi_{30} & \phi_{20} & -\phi_{23} & \phi_{023} \\ \phi_2 & -\phi_{20} & -\phi_{12} & \phi & \phi_{23} & -\phi_{012} & \phi_0 & \phi_{023} & -\phi_3 & -\phi_{123} & \phi_1 & \phi_{30} & -\phi_{0123} & -\phi_{10} & -\phi_{31} & \phi_{031} \\ \phi_3 & -\phi_{30} & \phi_{31} & -\phi_{23} & \phi & \phi_{031} & -\phi_{023} & \phi_0 & \phi_2 & -\phi_1 & -\phi_{123} & -\phi_{20} & \phi_{10} & -\phi_{0123} & -\phi_{12} & \phi_{012} \\ \phi_{10} & -\phi_1 & \phi_0 & \phi_{012} & -\phi_{031} & \phi & \phi_{12} & -\phi_{31} & \phi_{0123} & \phi_{30} & -\phi_{20} & \phi_{123} & -\phi_3 & \phi_2 & -\phi_{023} & \phi_{23} \\ \phi_{20} & -\phi_2 & -\phi_{012} & \phi_0 & \phi_{023} & -\phi_{12} & \phi & \phi_{23} & -\phi_{30} & \phi_{0123} & \phi_{10} & \phi_3 & \phi_{123} & -\phi_1 & -\phi_{031} & \phi_{31} \\ \phi_{30} & -\phi_3 & \phi_{031} & -\phi_{023} & \phi_0 & \phi_{31} & -\phi_{23} & \phi & \phi_{20} & -\phi_{10} & \phi_{0123} & -\phi_2 & \phi_1 & \phi_{123} & -\phi_{012} & \phi_{12} \\ \phi_{23} & \phi_{023} & -\phi_{123} & \phi_3 & -\phi_2 & -\phi_{0123} & -\phi_{30} & \phi_{20} & \phi & \phi_{12} & -\phi_{31} & \phi_0 & \phi_{012} & -\phi_{031} & -\phi_1 & -\phi_{10} \\ \phi_{31} & \phi_{031} & -\phi_3 & -\phi_{123} & \phi_1 & \phi_{30} & -\phi_{0123} & -\phi_{10} & -\phi_{12} & \phi & \phi_{23} & -\phi_{012} & \phi_0 & \phi_{023} & -\phi_2 & -\phi_{20} \\ \phi_{12} & \phi_{012} & \phi_2 & -\phi_1 & -\phi_{123} & -\phi_{20} & \phi_{10} & -\phi_{0123} & \phi_{31} & -\phi_{23} & \phi & \phi_{031} & -\phi_{023} & \phi_0 & -\phi_3 & -\phi_{30} \\ \phi_{023} & \phi_{23} & \phi_{0123} & \phi_{30} & -\phi_{20} & \phi_{123} & -\phi_3 & \phi_2 & \phi_0 & \phi_{012} & -\phi_{031} & \phi & \phi_{12} & -\phi_{31} & -\phi_{10} & -\phi_1 \\ \phi_{031} & \phi_{31} & -\phi_{30} & \phi_{0123} & \phi_{10} & \phi_3 & \phi_{123} & -\phi_1 & -\phi_{012} & \phi_0 & \phi_{023} & -\phi_{12} & \phi & \phi_{23} & -\phi_{20} & -\phi_2 \\ \phi_{012} & \phi_{12} & \phi_{20} & -\phi_{10} & \phi_{0123} & -\phi_2 & \phi_1 & \phi_{123} & \phi_{031} & -\phi_{023} & \phi_0 & \phi_{31} & -\phi_{23} & \phi & -\phi_{30} & -\phi_3 \\ \phi_{123} & \phi_{0123} & \phi_{23} & \phi_{31} & \phi_{12} & \phi_{023} & \phi_{031} & \phi_{012} & \phi_1 & \phi_2 & \phi_3 & -\phi_{10} & -\phi_{20} & -\phi_{30} & \phi & -\phi_0 \\ \phi_{0123} & \phi_{123} & -\phi_{023} & -\phi_{031} & -\phi_{012} & -\phi_{23} & -\phi_{31} & -\phi_{12} & -\phi_{10} & -\phi_{20} & -\phi_{30} & \phi_1 & \phi_2 & \phi_3 & -\phi_0 & \phi \end{array} \right)$$

In case  $\phi_Y$  are the unit vectors, this matrix is the sum of 16 matrices that form a representation of the algebra of  $16 \times 16$  matrices  $\{e_Y\}$  with real entries:  $M_{16}(\mathbb{R})$ .

The multiplication of two arbitrary multivectors  $\Psi = s + v + b + r + t + q$  yields

$$\Psi_\alpha \Psi_\beta = \Psi_\alpha \cdot \Psi_\beta + \Psi_\alpha \wedge \Psi_\beta = \frac{1}{2}(\Psi_\alpha \Psi_\beta + \Psi_\beta \Psi_\alpha) + \frac{1}{2}(\Psi_\alpha \Psi_\beta - \Psi_\beta \Psi_\alpha) \quad (3)$$

with

$$\Psi_\alpha \cdot \Psi_\beta = s_{0\alpha}s_{0\beta} + v_{0\alpha}v_{0\beta} - \vec{v}_\alpha \cdot \vec{v}_\beta + \vec{b}_\alpha \cdot \vec{b}_\beta - \vec{r}_\alpha \cdot \vec{r}_\beta - \vec{t}_\alpha \cdot \vec{t}_\beta + t_{0\alpha}t_{0\beta} - q_{0\alpha}q_{0\beta}$$

$$\begin{aligned}
& + e_0(s_{0\alpha}v_{0\beta} + s_{0\beta}v_{0\alpha} - \vec{r}_\alpha \cdot \vec{t}_\beta - \vec{r}_\beta \cdot \vec{t}_\alpha) \\
& + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (s_{0\alpha}\vec{v}_\beta + s_{0\beta}\vec{v}_\alpha - t_{0\alpha}\vec{r}_\beta - t_{0\beta}\vec{r}_\alpha + \vec{b}_\alpha \times \vec{t}_\beta + \vec{b}_\beta \times \vec{t}_\alpha) \\
& + \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (s_{0\alpha}\vec{b}_\beta + s_{0\beta}\vec{b}_\alpha + q_{0\alpha}\vec{r}_\beta + q_{0\beta}\vec{r}_\alpha + \vec{v}_\alpha \times \vec{t}_\beta + \vec{v}_\beta \times \vec{t}_\alpha) \\
& + \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} (s_{0\alpha}\vec{r}_\beta + s_{0\beta}\vec{r}_\alpha + v_{0\alpha}\vec{t}_\beta + v_{0\beta}\vec{t}_\alpha - t_{0\alpha}\vec{v}_\beta - t_{0\beta}\vec{v}_\alpha - q_{0\alpha}\vec{b}_\beta - q_{0\beta}\vec{b}_\alpha) \\
& + \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} (s_{0\alpha}\vec{t}_\beta + s_{0\beta}\vec{t}_\alpha + v_{0\alpha}\vec{r}_\beta + v_{0\beta}\vec{r}_\alpha + \vec{v}_\alpha \times \vec{b}_\beta + \vec{v}_\beta \times \vec{b}_\alpha) \\
& + e_{123}(s_{0\alpha}t_{0\beta} + s_{0\beta}t_{0\alpha} + \vec{v}_\alpha \cdot \vec{r}_\beta + \vec{v}_\beta \cdot \vec{r}_\alpha) \\
& + e_{0123}(s_{0\alpha}q_{0\beta} + s_{0\beta}q_{0\alpha} - \vec{b}_\alpha \cdot \vec{r}_\beta - \vec{b}_\beta \cdot \vec{r}_\alpha) \tag{4}
\end{aligned}$$

$$\Psi_\alpha \wedge \Psi_\beta = \tag{5}$$