Appendix A
Examination of the Similarities Between Confined Light and a Particle

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§ Confined Light

This appendix will investigate a photon confined in perfectly reflecting resonator. It will be shown that such a confined photon exhibits many particle-like properties including rest mass, relativistic contraction and a moving wave pattern that is similar to de Broglie waves.

We will begin by examining a standing wave in a resonator as viewed from a frame of reference in which the resonator is moving.

§ First View: Counter Propagating Waves

A 1-D standing wave can be modeled as a superposition of right and left moving plane waves.

$$\psi = e^{i(k_0 x - \omega_0 t)} + e^{i(-k_0 x - \omega_0 t)}$$

where $k_0 = \omega_0/c$.

In the frame of reference where the resonator is at rest there are standing waves in the resonator set up by the counter propagating waves.

$$\begin{aligned}\psi &= e^{i(k_0 x - \omega_0 t)} + e^{i(-k_0 x - \omega_0 t)} \\ &= (e^{ik_0 x} + e^{-ik_o x})e^{-i\omega_o t} \\ &= 2\cos(k_0 x)e^{-i\omega_o t}\end{aligned}$$

In the frame of reference where the resonator is moving to the right with velocity v, we have counter propagating waves with different frequencies, because the waves have been doppler shifted: $\omega_R = \gamma(1+\beta)\omega_0$ and $\omega_L = \gamma(1-\beta)\omega_0$, where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The wave then is given by

$$\psi = e^{i(k_R x - \omega_R t)} + e^{i(k_L x - \omega_L t)}$$

where $k_L = -\omega_L/c$ and $k_R = \omega_R/c$.

Define ω_+ and ω_- as follows

$$\begin{aligned}\omega_+ &\equiv \tfrac{1}{2}(\omega_R + \omega_L) \\ &= \tfrac{1}{2}(\gamma(1+\beta)\omega_0 + \gamma(1-\beta)\omega_0) \\ &= \gamma\omega_0 \\ \omega_- &\equiv \tfrac{1}{2}(\omega_R - \omega_L) \\ &= \tfrac{1}{2}(\gamma(1+\beta)\omega_0 - \gamma(1-\beta)\omega_0) \\ &= \gamma\beta\omega_0\end{aligned}$$

Now define k_+ and k_- in a similar way.

$$k_+ \equiv \tfrac{1}{2}(k_R + k_L) = \frac{1}{2c}(\omega_R - \omega_L) = \frac{\omega_-}{c} = \gamma\beta\frac{\omega_0}{c}$$

$$k_- \equiv \tfrac{1}{2}(k_R - k_L) = \frac{1}{2c}(\omega_R + \omega_L) = \frac{\omega_+}{c} = \gamma\frac{\omega_0}{c}$$

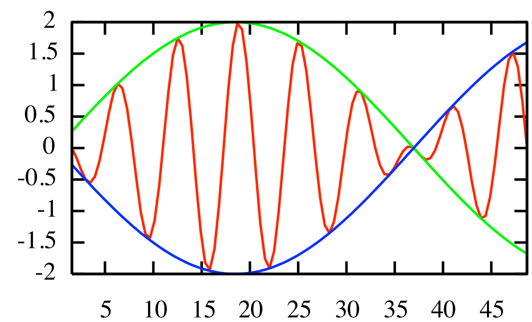
Note that

$$\begin{aligned}k_L &= k_+ - k_- \\ k_R &= k_+ + k_-\end{aligned} \qquad \text{AND} \qquad \begin{aligned}\omega_L &= \omega_+ - \omega_- \\ \omega_R &= \omega_+ + \omega_-\end{aligned}$$

Now we can write the wave as

$$\begin{aligned}\psi &= e^{i(k_L x - \omega_L t)} + e^{i(k_R x - \omega_R t)} \\ &= e^{i((k_+ - k_-)x - (\omega_+ - \omega_-)t)} + e^{i((k_+ + k_-)x - (\omega_+ + \omega_-)t)} \\ &= \left[e^{-i(k_- \ x - \omega_- \ t)} + e^{i(k_- \ x - \omega_- \ t)}\right] e^{i(k_+ x - \omega_+ t)} \\ &= 2\cos(k_- \ x - \omega_- \ t)e^{i(k_+ x - \omega_+ t)}\end{aligned}$$

The imaginary part of this is graphed below (in red) for $\beta = 0.085$ at $t = 0$.



This is a product of two traveling waves. We can compute wavelengths and velocities of these two parts.

$$\begin{aligned}v_+ &= \frac{\omega_+}{k_+} = \frac{\omega_+}{\omega_-/c} = \frac{c}{\beta} = \frac{c^2}{v} \\ v_- &= \frac{\omega_-}{k_-} = \frac{\omega_-}{\omega_+/c} = \beta c = v \\ \lambda_+ &= \frac{2\pi}{k_+} = \frac{2\pi c}{\gamma\beta\omega_0} = \frac{\lambda_0}{\gamma\beta} \\ \lambda_- &= \frac{2\pi}{k_-} = \frac{2\pi c}{\gamma\omega_0} = \frac{\lambda_0}{\gamma}\end{aligned}$$

where $\lambda_0 = \frac{2\pi c}{\omega_0}$ is the wavelength in the rest frame of the resonator.

First let us consider the "-" part of the wave. First we note that this part of the wave moves with the velocity of the resonator. Second we see that the wavelength has shrunk by a factor of γ relative to the wavelength in the rest frame. Thus the same number of wavelengths will fit in the similarly length-contracted resonator. Thus the cos() standing wave pattern has shrunk to fit the moving resonator and moves with the resonator.

Now consider the "+" part of the wave. This part of the wave moves with a velocity c^2/v. Which is the same as the phase velocity of a de Broglie plane wave for a massive particle: $\psi_d = e^{ipx/\hbar}e^{-iEt/\hbar}$.

$$\longrightarrow \ v_{\text{phase}} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p} = \frac{\gamma mc^2}{\gamma mv} = \frac{c^2}{v}$$

We can also see that the wavelength for the de Broglie plane wave: $\lambda_d = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{Ev/c^2} = \frac{2\pi\hbar c}{E\beta} = \frac{2\pi\hbar c}{\gamma E_0\beta}$ is also the same, if we assume that there is a single photon in the resonator and thus that the energy in the rest frame is $E_0 = \hbar\omega_0$. Since

$$\lambda_+ = \frac{2\pi}{k_+} = \frac{2\pi c}{\gamma\beta\omega_0} = \frac{2\pi\hbar c}{\gamma\beta E_0}$$

Thus we see that the "+" part of the resonator wave has the wavelength and phase velocity of a de Broglie plane wave of a massive particle with a rest energy equal to the energy of the photon in the resonator.

§ General From

In the rest frame of a general standing wave the amplitude of the wave is given by

$$\psi'(x',y',z',t') = f(x',y',z')e^{-i\omega_0 t'}$$

Where it is understood that the physical wave is the real part of the complex wave ψ. Note that the amplitude function $f()$ is a real valued function.

We will assume that the energy of this wave is

$$E_0 = \hbar\omega_0.$$

We will also consider this energy in the rest frame of the wave, divided by c^2, to be the mass of the system:

$$m \equiv \frac{E_0}{c^2} = \frac{\hbar\omega_0}{c^2}.$$

Since the wave is a standing wave the total momentum is zero: $p_0 = 0$.

In a General Frame

We want to know what the standing wave will look like in a frame in which the rest frame of the standing wave is moving in the positive x direction, with a velocity v. We can find this, if we assuming that the amplitude of the wave at a given space time point is the same in each frame, so that

$$\psi(x,y,z,t) = \psi'(x',y',z',t')$$

with x' and t' related to x and t via the following Lorentz transformation, while $y' = y$ and $z' = z$.

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} \gamma(ct-\beta x) \\ \gamma(x-\beta ct) \end{bmatrix}$$

So that

$$\begin{aligned} \psi(x,y,z,t) &= \psi'(x',y',z',t') \\ &= \psi'(\ \gamma(x{-}\beta ct), y, z,\ \gamma(ct{-}\beta x)/c\ ) \\ &= f(\gamma(x-\beta ct), y, z)e^{-i\omega_0\gamma(ct-\beta x)/c} \\ &= f(\gamma(x-vt), y, z)e^{i\omega_0\gamma\beta x/c}e^{-i\omega_0\gamma t} \\ &= f(\gamma(x-vt), y, z)e^{i(kx-\omega t)} \end{aligned}$$

In the last line we used the following definition.

$$\begin{aligned} \omega &\equiv \gamma\omega_0 \\ k &\equiv \gamma\beta\omega_0/c \end{aligned}$$

As was the case with the counter propagating plane waves, the wave function in the general frame is manifestly in the form of a plane wave ($e^{i(kx-\omega t)}$), with wavelength and velocity of a de Broglie wave, modulated with an standing wave pattern (f) that moves in the positive x direction with velocity v. In addition the characteristic length of the standing wave pattern has been length contracted in the x direction by the factor γ compared with the length in the rest frame. For examples suppose that there are two features in the standing wave, one at the position $x' = a$ and the other at the position $x' = b$. The distance between these features is $L = b - a$. The features could for example be two null points in the standing wave. These two features will also exist in the general frame, though they are moving. Let x_a and x_b the location of these features, then we know that

$$\begin{aligned} \gamma(x_a - vt) &= a \\ \gamma(x_b - vt) &= b \end{aligned}$$

Solving these two equations for $x_b - x_a$ we find that

$$L' = x_b - x_a = \frac{b-a}{\gamma} = \frac{L}{\gamma}$$

So the distance between the features has been contracted by a factor γ.

We also see, as in the case of counter propagating waves, that one part of the wave moves with the velocity v and the other moves with the velocity

$$\frac{\omega}{k} = \frac{\gamma\omega_0}{\gamma\beta\omega_0/c} = \frac{c}{\beta} = \frac{c^2}{v}$$

Energy and Momentum

We can also find the energy and momentum in the new frame, using the relativistic transformation of energy and momentum.

$$\begin{aligned} \begin{bmatrix} E \\ pc \end{bmatrix} &= \begin{bmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} E' \\ p'c \end{bmatrix} \\ &= \begin{bmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} E_0 \\ p_0 c \end{bmatrix} \\ &= \begin{bmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} \hbar\omega_0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \gamma\hbar\omega_0 \\ \gamma\beta\hbar\omega_0 \end{bmatrix} \\ &= \begin{bmatrix} \hbar\omega \\ \hbar kc \end{bmatrix} \end{aligned}$$

We see that the frequency and wavenumber of the plane wave part of the wave are proportional to the energy and momentum of the wave.

$$E = \hbar\omega$$

and

$$p = \hbar k$$

in accordance with de Broglie waves.

Using the definition $m = \hbar\omega_0/c^2$, we consider the mass of the light in our resonator to be equal to the energy in the rest frame divided by c^2. Thus we can rewrite the above as.

$$E = \gamma mc^2$$

and

$$p = \gamma mv$$