PACS №: 02.40.Re; 03.65.Vf
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Topology and Electromagnetism

Contents

Abstract

We attempt to show the fundamental explanatory nature of the topological description of solitons, instantons and the Aharonov-Bohm effect — and hence electromagnetism. In the case of electromagnetism we have shown elsewhere that, given a Yang-Mills description, electromagnetism can, and should be extended, in accordance with the topology with which the electromagnetic fields are associated. This approach has major implications.

1. Introduction

Topology addresses those properties, often associated with *invariant qualities*, which are not altered by continuous deformations. Objects are topologically equivalent, or *homeomorphic*, if one object can be changed into another by bending, stretching, twisting, or any other continuous deformation or mapping. Continuous deformations are allowed, but prohibited are foldings which bring formerly distant points into direct contact or overlap, and cutting – unless followed by a regluing, reestablishing the preexisting relationships of continuity.

The continuous deformations of topology are commonly described in differential equation form and the quantities conserved under the transformations commonly described by differential equations exemplifying an algebra describing operations which preserve that algebra. Evariste Galois (1811–1832) first gave the criteria that an algebraic equation must satisfy in order to be solvable by radicals. This branch of mathematics came to be known as Galois or *group theory.*

Beginning with G.W. Leibniz in the 17th, L. Euler in the 18th, B. Reimann, J.B. Listing and A.F. Möbius in the 19th and H. Poincaré in the 20th centuries, *"analysis situs"* (Riemann) or *"topology"* (Listing) has been used to provide answers to questions concerning what is most fundamental in physical explanation. That question itself implies the question concerning what mathematical structures one uses with confidence to adequately "paint" or describe physical models built from empirical facts. For example, differential equations of motion cannot be fundamental, because they are dependent on boundary conditions which must be justified – usually by group theoretical considerations. Perhaps, then, group theory is fundamental.

Here we mean the kind of groups addressed in Yang-Mills theory, which are *continuous* groups (as opposed to *discrete* groups). Unlike discrete groups, continuous groups contain an infinite number of elements and can be differentiable or analytic [1].

Group theory certainly offers an austere shorthand for fundamental transformation rules. But it appears to the present writer that the final judge of whether a mathematical group structure can, or cannot, be applied to a physical situation is the topology of that physical situation. Topology dictates and justifies the group transformations.

So for the present writer, the answer to the question of what is the most fundamental physical description is that it is a description of the topology of the situation. With the topology known, the group theory description is justified and equations of motion can then also be justified and defined in specific differential equation form. If there is a requirement for an understanding more basic than the topology of the situation, then all that is left is verbal description of visual images. So we commence an examination of electromagnetism under the assumption that topology defines group transformations and the group transformation rules justify the algebra underlying the differential equations of motion.

Differential equations, or a set of differential equations, describe a *system* and its evolution. Group symmetry principles summarize both invariances and the laws of nature independent of a system's specific dynamics. It is necessary that the symmetry transformations be continuous or specified by a set of parameters which can be varied continuously. The symmetry of continuous transformations leads to conservation laws.

There are a variety of special methods used to solve ordinary differential equations. It was Sophus Lie (1842–1899) in the 19^{th} century who showed that all the methods are special cases of integration procedures which are based on the invariance of a differential equation under a continuous group of symmetries. These groups became known as Lie groups.

If a topological group is a group and also a topological space in which group operations are continuous, then *Lie groups* are topological groups which are also analytic manifolds on which the group operations are analytic.

In the case of *Lie algebras*, the parameters of a product are analytic functions of the parameters of each factor in the product. For example, $L(\gamma) = L(\alpha)L(\beta)$ where $\gamma = f(\alpha, \beta)$. This guarantees that the group is differentiable. The Lie groups used in Yang-Mills theory are *compact groups*, i.e., the parameters range over a closed interval. A symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions [2]. In other words, there is an invariance of a differential equation under a transformation of independent and dependent variables. This invariance results in a diffeomorphism on the space of independent and dependent variables, permitting the mapping of solutions to solutions [3].

The relationship was made more explicit by Emmy (Amalie) Noether (1882–1935) in theorems now known as *Noether's theorems* [4], which related symmetry groups of a variational integral to properties of its associated Euler-Lagrange equations.

The most important consequences of this relationship are that

(i) conservation of energy arises from invariance under a group of time translations;

(ii) conservation of linear momentum arises from invariance under (spatial) translational groups;

(iii) conservation of angular momentum arises from invariance under (spatial) rotational groups;

(iv) conservation of charge arises from invariance under change of phase of the wave function of charged particles.

Conservation and group symmetry laws have been vastly extended to other systems of equations, e.g., the standard model of modern high energy physics, and also, of importance to the present interest: soliton equations. For example, the *Korteweg de Vries* "soliton" equation [5] yields a symmetry algebra spanned by the four vector fields of

(i) space translation;

(ii) time translation;

(iii) Galilean translation;

(iv) scaling.

For some time, the present writer has been engaged in showing that the space-time topology defines electromagnetic field equations [6] — whether the fields be of force or of phase. That is to say, the premise of this enterprise is that a set of field equations are only valid with respect to a set defined topological description of the physical situation. In particular, the writer has addressed demonstrating that the $\boldsymbol{A}_\mu$ potentials, $\mu = 0, 1, 2, 3$, are not just a mathematical convenience, but — *in certain well-defined situations* — are measurable, i.e., physical. Those situations in which the $\boldsymbol{A}_\mu$ potentials are measurable possess a topology, the transformation rules of which are describable by the $SU(2)$ group [2] or higher-order groups; and those situations in which the $\boldsymbol{A}_\mu$ potentials are not measurable possess a topology, the transformation rules of which are describable by the $U(1)$ group [2].

Historically, electromagnetic theory was developed for situations described by the $U(1)$ group. The dynamic equations describing the transformations and interrelationships of the force field are the well known Maxwell equations, and the group algebra underlying these equations is $U(1)$. There was a need to extend these equations to describe $SU(2)$ situations and to derive equations whose underlying algebra is $SU(2)$ These two formulations are shown

Table 1.

	$U(1)$ Symmetry Form (Traditional Maxwell Equations)	$SU(2)$ Symmetry Form
Gauss' Law	$\nabla \bullet \boldsymbol{E} = \boldsymbol{J}_0$	$\nabla \bullet \boldsymbol{E} = \boldsymbol{J}_0 - iq(\boldsymbol{A} \bullet \boldsymbol{E} - \boldsymbol{E} \bullet \boldsymbol{A})$
Ampere's Law	$\dfrac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} - \boldsymbol{J} = 0$	$\dfrac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} - \boldsymbol{J} + iq[\boldsymbol{A}_0, \boldsymbol{E}] - iq(\boldsymbol{A} \times \boldsymbol{B} - \boldsymbol{B} \times \boldsymbol{A}) = 0$
	$\nabla \bullet \boldsymbol{B} = 0$	$\nabla \bullet \boldsymbol{B} + iq(\boldsymbol{A} \bullet \boldsymbol{B} - \boldsymbol{B} \bullet \boldsymbol{A}) = 0$
Faraday's Law	$\nabla \times \boldsymbol{E} + \dfrac{\partial \boldsymbol{B}}{\partial t} = 0$	$\nabla \times \boldsymbol{E} + \dfrac{\partial \boldsymbol{B}}{\partial t} + iq[\boldsymbol{A}_0, \boldsymbol{B}] = iq(\boldsymbol{A} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{A}) = 0$

in Table 1. Table 2 shows the electric charge density, ρ_e, the magnetic charge density, ρ_m, the electric current density, g_e, the magnetic current density, g_m, the electric conductivity, σ, and the magnetic conductivity, s.

In the following sections, four topics are addressed: The mathematical entities, or waves, called *solitons*; the mathematical entities called *instantons*; a beam — an electromagnetic wave — which is *polarization modulated over a set sampling interval*; and the *Aharonov-Bohm effect*. Our intention is to show that these entities, waves or effects, can only be adequately characterized and differentiated, and thus understood, by using topological characterizations. Once characterized, the way becomes open for control or engineering of these entities, waves and effects.

2. Solitons

A soliton is a solitary wave which preserves its shape and speed in a collision with another solitary wave [7,8]

Soliton solutions to differential equations require complete integrability and integrable systems conserve geometric features related to symmetry. Unlike the equations of motion for conventional Maxwell theory, which are solutions of $U(1)$ symmetry systems, solitons are solutions of $SU(2)$ symmetry systems. These notions of group symmetry are more fundamental than differential equation descriptions. Therefore, although a complete exposition is beyond the scope of the present review, we develop some basic concepts in order to place differential equation descriptions within the context of group theory.

Within this context, *ordinary differential equations are viewed as vector fields on manifolds or configuration spaces* [2]. For example, Newton's equations are second order differential equations describing smooth curves on Riemannian manifolds. Noether's theorem [4] states that a diffeomorphism, ϕ, of a Riemannian manifold, C, indices a diffeomorphism, $D\phi$, of its tangent bundle, TC.

If ϕ is a symmetry of Newton's equations, then $D\phi$ preserves the Lagrangian, i.e.,

$$\boldsymbol{L} \circ D\phi = \boldsymbol{L}\,.$$

As opposed to equations of motion in conventional Maxwell theory, *soliton flows are Hamiltonian flows.* Such Hamiltonian functions define *symplectic structures* for which there is an absence of *local* invariants but an infinite dimensional group of diffeomorphisms which preserve *global* properties. *Symplectic topology* is the study of the global phenomena of symplectic symmetry. Symplectic symmetry structures have no local invariants. This is a subfield of topology, for example [9]. In the case of solitons, the global properties are those permitting the matching of the nonlinear and dispersive characteristics of the medium through which the wave moves.

In order to achieve this match, two linear operators, L and A, are postulated associated with a partial differential equation (PDE). The two linear operators are known as the *Lax pair.* The operator L is defined by:

$$L = \frac{\partial^2}{\partial x^2} + u(x,t),$$

with a related eigenproblem:

$$L\psi + \lambda\psi = 0. \tag{1}$$

Table 2.

$U(1)$ Symmetry Form (Traditional Maxwell Equations)	$SU(2)$ Symmetry Form
$\rho_e = J_0$	$\rho_e = \boldsymbol{J}_0 - iq(\boldsymbol{A} \bullet \boldsymbol{E} - \boldsymbol{E} \bullet \boldsymbol{A}) = \boldsymbol{J}_0 + q\boldsymbol{J}_z$
$\rho_m = 0$	$\rho_m = -iq(\boldsymbol{A} \bullet \boldsymbol{B} - \boldsymbol{B} \bullet \boldsymbol{A}) = -iq\boldsymbol{J}_y$
$g_e = \boldsymbol{J}$	$g_e = iq[\boldsymbol{A}_0, \boldsymbol{E}] - iq(\boldsymbol{A} \times \boldsymbol{B} - \boldsymbol{B} \times \boldsymbol{A}) + \boldsymbol{J} = iq[\boldsymbol{A}_0, \boldsymbol{E}] - iq\boldsymbol{J}_x + \boldsymbol{J}$
$g_m = 0$	$g_m = iq[\boldsymbol{A}_0, \boldsymbol{B}] - iq(\boldsymbol{A} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{A}) = iq[\boldsymbol{A}_0, \boldsymbol{B}] - iq\boldsymbol{J}_z$
$\sigma = \boldsymbol{J}/\boldsymbol{E}$	$\sigma = \dfrac{\{iq[\boldsymbol{A}_0, \boldsymbol{E}] - iq(\boldsymbol{A} \times \boldsymbol{B} - \boldsymbol{B} \times \boldsymbol{A}) + \boldsymbol{J}\}}{\boldsymbol{E}} = \boldsymbol{J}\dfrac{\{iq[\boldsymbol{A}_0, \boldsymbol{E}] - iq\boldsymbol{J}_x + \boldsymbol{J}\}}{\boldsymbol{E}}$
$s = 0$	$s = \dfrac{\{iq[\boldsymbol{A}_0, \boldsymbol{B}] - iq(\boldsymbol{A} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{A})\}}{\boldsymbol{H}} = \dfrac{\{iq[\boldsymbol{A}_0, \boldsymbol{B}] - iq\boldsymbol{J}_z\}}{\boldsymbol{H}}$

The temporal evolution of ψ is defined as:

$$\psi_t = -A\psi, \qquad (2)$$

with the operator of the form:

$$A = a_0 \frac{\partial^n}{\partial x^n} + a_1 \frac{\partial^{n-1}}{\partial x^{n-1}} + ... + a_n,$$

where a_0 is a constant and the n coefficients a_i are functions of x and t. Differentiating (1) gives:

$$L_t\psi + L\psi_t = -\lambda_t\psi - \lambda\psi_t\,.$$

Inserting (2):

$$L\psi_t = -LA\psi\,,$$

or

$$\lambda\psi_t = AL\psi\,.$$

Using (2) again:

$$[L, A] = LA - AL = L_t + \lambda_t\,, \qquad (3)$$

and for a time-independent λ :

$$[L, A] = L_t\,.$$

This equation provides a method for finding A.

Translating the above into a group theory formulation: in order to relate the three major soliton equations to group theory it is necessary to examine the *Lax equation* [10] (3) above as a the *zero-curvature condition (ZCC)*. The *ZCC* expresses the flatness of a connection by the commutation relations of the covariant derivative operators [11] and in terms of the Lax equation is:

$$L_t - A_x - [L, A] = 0,$$

or [11]:

$$\left[\frac{\partial}{\partial x} - L\frac{\partial}{\partial t} - A\right] = 0,$$

or:

$$\left(\frac{\partial}{\partial x} - L\right)_t = \left[A\frac{\partial}{\partial x} - L\right].$$

More recently, Palais [11] showed that the generic cases of *soliton — the Korteweg de Vries Equation (KDV)*, the *Nonlinear Schrodinger Equation (NLS)*, the *Sine-Gordon Equation (SGE)* — can be given an *SU(2)* formulation. In each of the three cases considered below, V is a one-dimensional space that is embedded in the space of off-diagonal complex matrices, $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ and in each case $L(u) = \boldsymbol{a}\lambda + u$, where u is a potential, λ is a complex parameter, and $\boldsymbol{a}$ is the constant, diagonal, trace zero matrix

$$\mathbf{a} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

The matrix definition of $\boldsymbol{a}$ links these equations to an $SU(2)$ formulation. (Other matrix definitions of $\boldsymbol{a}$ could, of course, link $\boldsymbol{a}$ to higher group symmetries.)

To carry out this objective, an inverse scattering theory function is defined [12]:

$$B(\xi) = \sum_{n=1}^{N} c_n^2 \exp\left[-k_n \xi\right] + \frac{1}{2\pi} \int_{-\infty}^{+\infty} b(k) \exp\left[ik\xi\right] dk,$$

where $-k_1^2, \ldots, -k_n^2$ *are discrete eigenvalues of* u, $c_1, \ldots, c_N$ *are normalizing constants, and* $b(k)$ *are reflection coefficients.*

Therefore, in a ***first case*** (the KdV), if $u(x) = \begin{pmatrix} 0 & q(x) \\ -1 & 0 \end{pmatrix}$ and

$$B(u) = \boldsymbol{a}\lambda^3 + u\lambda^2 + \begin{pmatrix} \frac{i}{2}q & \frac{i}{2}q_x \\ 0 & -\frac{i}{2}q \end{pmatrix} \lambda + \begin{pmatrix} \frac{q_x}{4} & \frac{-q^2}{2} \\ \frac{q}{2} & \frac{-q_x}{4} \end{pmatrix},$$

and the ZCC (*Lax equation*) is satisfied if and only if q satisfies the KdV in the form $q_t = -\frac{1}{4}(6qq_x + q_{xxx})$.

In a ***second case*** (the NLS), if $u(x) = \begin{pmatrix} 0 & q(x) \\ -\overline{q}(x) & 0 \end{pmatrix}$ and

$$B(u) = \boldsymbol{a}\lambda^3 + u\lambda^2 + \begin{pmatrix} \frac{i}{2}|q|^2 & \frac{i}{2}q_x \\ -\frac{i}{2}\overline{q}(x) & -\frac{i}{2}|q|^2 \end{pmatrix},$$

then ZCC (*Lax equation*) is satisfied if and only if *q(x,t)* satisfies the NLS in the form $q_t = \frac{i}{2}\left(q_{xx} + 2|q|^2 q\right)$.

In a ***third case*** (the SGE), if

$$u(x) = \begin{pmatrix} 0 & -\frac{q_x(x)}{2} \\ \frac{q_x(x)}{2} & 0 \end{pmatrix}$$

and

$$B(u) = \frac{i}{4\pi} \begin{pmatrix} \cos[q(x)] & \sin[q(x)] \\ \sin[q(x)] & -\cos[q(x)] \end{pmatrix},$$

then ZCC (*Lax equation*) is satisfied if and only if q satisfies the SGE in the form $q_t = \sin[q]$.

With the connection of PDEs, and especially soliton forms, to group symmetries established, then one can conclude that *if* the Maxwell equation of motion which includes electric *and* magnetic conductivity is in soliton (SGE) form, the group symmetry of the Maxwell field is $SU(2)$. Furthermore, because solitons define Hamiltonian flows, their energy conservation is due to their *symplectic structure*.

In order to clarify the difference between conventional Maxwell theory which is of $U(1)$ symmetry, and Maxwell theory extended to $SU(2)$ symmetry, we can describe both in terms of mappings of a field $\psi(x)$. In the case of $U(1)$ Maxwell theory, a mapping $\psi \to \psi'$ is:

$$\psi(x) \to \psi'(x) = \exp[i\boldsymbol{a}(x)]\psi(x)\,,$$

where $\boldsymbol{a}(x)$ is the conventional vector potential. However, in the case of $SU(2)$ extended Maxwell theory, a mapping $\psi \to \psi'$ is:

$$\psi(x) \to \psi'(x) = \exp[i\boldsymbol{S}(x)]\psi(x),$$

where $\boldsymbol{S}(x)$ is the action and an element of an $SU(2)$ field defined:

$$\boldsymbol{S}(x) = \int \boldsymbol{A} dx,$$

and $\boldsymbol{A}$ is *the matrix form of the vector potential.* Therefore we see the necessity *to adopt a matrix formulation of the vector potential when addressing* $SU(2)$ *forms of Maxwell theory.*

3. Instantons

Instantons [12] correspond to the minima of the Euclidean action and are pseudo-particle solutions [13] of $SU(2)$ Yang-Mills equations in Euclidean 4 space [14]. A complete construction for any Yang-Mills group is also available [15]. In other words:

> "It is reasonable... to ask for the determination of the classical field configurations in Euclidean space which minimize the action, subject to appropriate asymptotic conditions in 4-space. These classical solutions are the instantons of the Yang-Mills theory" [16].

In the light of the intention of the present writer to introduce topology into electromagnetic theory, I quote further:

> "If one were to search ab initio for a non-linear generalization of Maxwell's equation to explain elementary particles, there are various symmetry group properties one would require. These are
> (i) *external symmetries* under the Lorentz and Poincaré groups and under the conformal group if one is taking the rest-mass to be zero,
> (ii) *internal symmetries* under groups like $SU(2)$ or $SU(3)$ to account for the known features of elementary particles,
> (iii) *covariance* or the ability to be coupled to gravitation by working on curved space-time" [17].

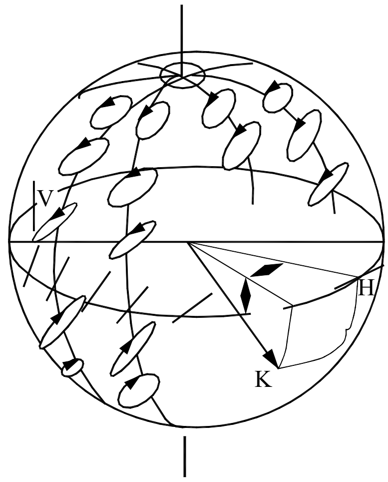
The present writer applied the instanton concept in electromagnetism for the following two reasons: (1) in some sense, the instanton, or pseudoparticle, is a compactification of degrees of freedom due to the particle's boundary conditions; and (2) the instanton, or pseudoparticle, then exhibits the behavior (the transformation or symmetry rules) of a high energy particle, but without the presence of high energy, i.e., the pseudoparticle shares certain behavioral characteristics in common (shares transformation rules, hence symmetry rules in common) with a particle of much higher energy.

Therefore, the present writer suggested [18] that the Mikhailov effect [19], and the Ehrenhaft effect (1879–1952), which address demonstrations exhibiting *magnetic charge-like* behavior, are examples of instanton or pseudoparticle behavior. Stated differently: (1) the instanton shows that there are ways, other than possession of high energy, to achieve high symmetry states; and (2) symmetry dictates behavior.

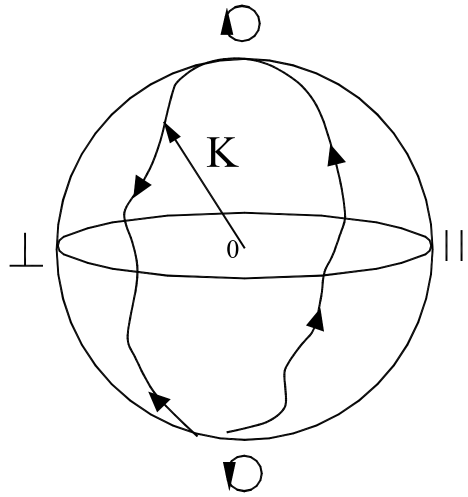
4. Polarization Modulation Over a Set Sampling Interval [20]

It is well-known that all static polarizations of a beam of radiation, as well as all static rotations of the axis of that beam can be represented on a Poincaré sphere [21] (Fig. 1A). A vector can be centered in the middle of the sphere and pointed to the underside of the surface of the sphere at a location on the surface which represents the *instantaneous* polarization and rotation angle of a beam. Causing that vector to trace a trajectory over time on the surface of the sphere represents a *polarization modulated* (and *rotation modulated*) beam (Fig. 1B). If, then, the beam is sampled by a device at a rate which is less than the rate of modulation, then the sampled output from the device will be a condensation of *two components* of the wave, which are continuously changing with respect to each other, into *one snapshot* of the wave, at *one location* on the surface of the sphere and *one instantaneous polarization and axis rotation.* Thus, from the viewpoint of a device sampling at a rate less than the modulation rate, a two-to-one mapping (over time) has occurred, which is the signature of an $SU(2)$ field.

The modulations which result in trajectories on the sphere are infinite in number. Moreover, those modulations, at a rate of multiples of 2π greater than 1, which result in the return to a single location on the sphere at a frequency of exactly 2π, will all be detected by the device sampling at a rate of 2π as the same. In other words, the device cannot detect what kind of simple or complicated trajectory was performed between departure from, and arrival at, the same location on the sphere. To the relatively slowly sampling device, the fast modulated beam can have "internal energies" quite unsuspected.



(a) A Poincaré sphere representation of wave polarization and rotation.



(b) A Poincaré sphere representation of signal polarization (longitudinal axis) and polarization rotation (latitudinal axis). A representational trajectory of polarization/rotation modulation is shown by changes in the vector centered at the center of the sphere and pointing at the surface. Waves of various polarization modulations $\partial\phi^n/\partial t^n$, can be represented as trajectories on the sphere. The case shown is an arbitrary trajectory repeating 2π. After Ref [23].

Fig. 1.

We can say that such a static device is *a* $U(1)$ *unipolar, set rotational axis*, sampling device and the fast polarization (and rotation) modulated beam is a *multipolar, multirotation axis,* $SU(2)$ *beam.* The reader may ask: how many situations are there in which a sampling device, at set unvarying polarization, samples at a slower rate than the modulation rate of a radiated beam? The answer is that there is an infinite number, because from the point of the view of the writer, nature is set up to be that way [22]. For example, the period of modulation can be faster than the electronic or vibrational or dipole relaxation times of any atom or molecule. In other words, *pulses or wave packets* (which, in temporal length, constitute the sampling of a continuous wave, continuously polarization and rotation modulated, but sampled only over a temporal length between arrival and departure time at the instantaneous polarization of the sampler of set polarization and rotation — in this case an electronic or vibrational state or dipole) have an internal modulation at a rate greater than that of the relaxation or absorption time of the electronic or vibrational state.

If a spin matrix, $\boldsymbol{A}$, is defined:

$$\boldsymbol{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \det \boldsymbol{A} = 1\,,$$

then the two transformations, (Fig. 1A), are:

$$\boldsymbol{A} = \begin{pmatrix} \xi' & \xi \\ \eta' & \eta \end{pmatrix}, \qquad (4)$$

which means that the spin matrix of a composition is given by the product of the spin matrix of the factors. Any transformation of the (4) form is linear and real and leaves the form $W^2 - X^2 - Y^2 - Z^2$ invariant.

Furthermore, there is a unimodular condition:

$$\alpha\delta - \beta\gamma = 1\,,$$

and the matrix $\boldsymbol{A}$ has the inverse:

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix},$$

which means that the spin matrix $\boldsymbol{A}$ and its inverse $\boldsymbol{A}^{-1}$ gives rise to the same transformation of ζ even although they define different spin transformations. Due to the unimodular condition, the $\boldsymbol{A}$ spin-matrix is unitary or:

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^*$$

where $\boldsymbol{A}^*$ is the conjugate transpose of $\boldsymbol{A}$.

The consequence of these relations is that *every proper* 2π *rotation on* $\boldsymbol{S}^+$ *— in the present instance the Poincaré sphere — corresponds to precisely two unitary spin rotations.* As every rotation on the Poincaré sphere corresponds to a polarization/rotation *modulation*, then *every proper* 2π *polarization-rotation modulation corresponds to precisely two unitary spin rotations.*

Using this algebraic formalism, the Poincaré vector — *and its direction of change (up to sign ambiguity)* — can be represented. A real tangent vector $\boldsymbol{L}$ of $\boldsymbol{S}^+$ at P is defined:

$$\boldsymbol{L} = \frac{\lambda\partial}{\partial\zeta} + \frac{\lambda^*\partial}{\partial\zeta^*}\,,$$

where λ is some expression in ξ, η. With the choice $\lambda = -\left(\frac{1}{\sqrt{2}}\right)\eta^{-2}$:

$$\boldsymbol{L} = \left(\frac{1}{\sqrt{2}}\right)\left[\eta^{-2}\left(\frac{\partial}{\partial\zeta}\right) + \eta^{*-2}\left(\frac{\partial}{\partial\zeta^*}\right)\right],$$

and thus knowing $\boldsymbol{L}$ at P (as an operator) means that the pair ξ, η is known completely up to sign, or, for any $f(\zeta, \zeta^*)$:

$$\frac{1}{\varepsilon_{\lim \varepsilon\to 0}}(fp' - fp) = \boldsymbol{L}f$$

Succinctly: the tangent vector $\boldsymbol{L}$ in the abstract space $\boldsymbol{S}^+$ (Poincaré sphere) corresponds to a tangent vector L in the coordinate-dependent representation S^+ of $\boldsymbol{S}^+$. L is a unit vector if and only if, K, the null vector corresponding to ξ, η, defines a point actually on $\boldsymbol{S}^+$. Therefore a plane of K and L can be defined by:

$$aK + bL,$$

and if $b > 0$, then a half-plane, Π, is defined bounded by K. K and L are both spacelike and orthogonal to each other. In the twistor formalism [24], Π and K are referred to as a *null flag* or a *flag*. The vector K is called the *flagpole*, its direction is the *flagpole direction* and the half-plane, Π , is the *flag plane*.

The major point here is that a polarization-rotation modulated wave can be represented as a periodic trajectory of polarization/rotation modulation on a Poincaré sphere, or a *spinorial object. A defining characteristic of a spinorial object is that it is not returned to its original state when rotated through an angle* 2π *about some axis, but only when rotated through* 4π. *We thus see that for the resultant to be rotated through* 2π *and returned to its original polarization state, the* ***operator*** *must be rotated through* 4π. *Thus a spinorial object (polarization/rotation modulated beams) exists in a different topological space from static polarized/rotated beams due to the additional degree of freedom provided by the polarization bandwidth, which does not exist prior to modulation.*

The relation to the electromagnetic field is as follows. The (antisymmetrical) inner product of two spin vectors can represented as:

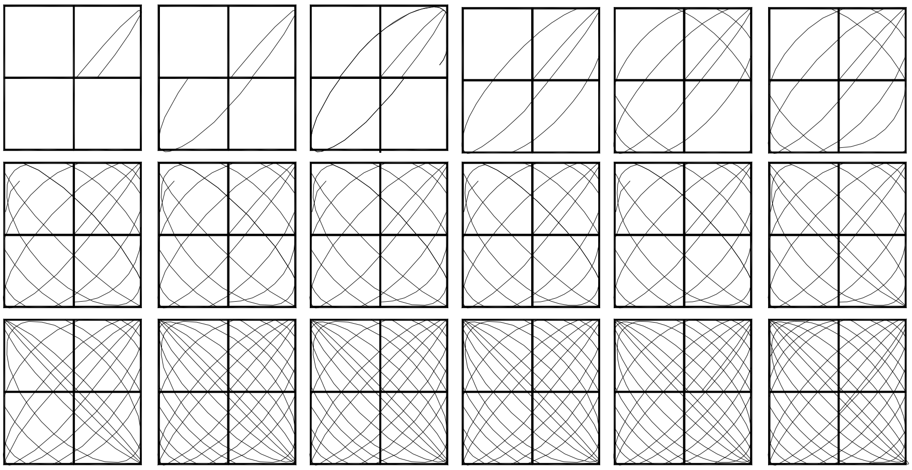
$$\{k_1, k_2\} = \varepsilon_{AB}k^Ak^B = -\{k_1, k_2\}\,,$$

where the ε (or the fundamental numerical metric spinors of second rank) are antisymmetrical:

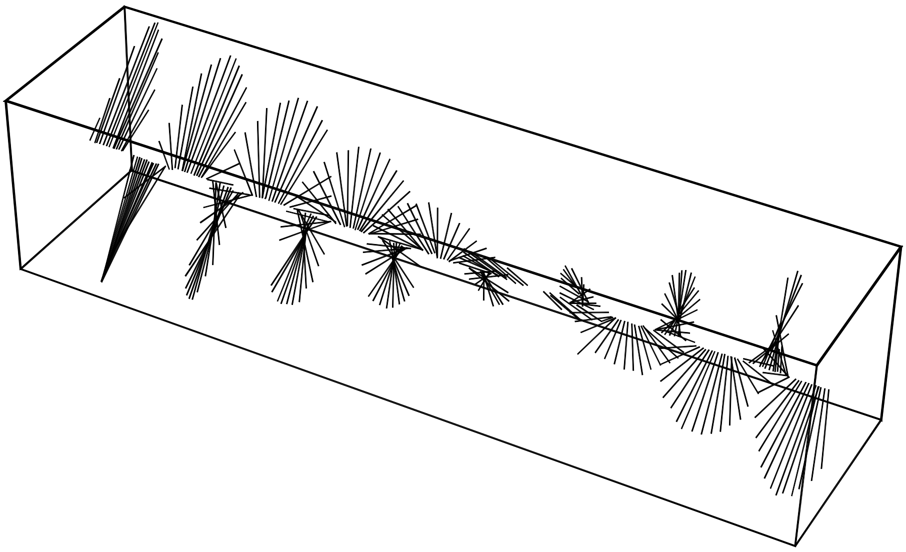
$$\varepsilon_{AB}\varepsilon^{CB} = -\varepsilon_{AB}\varepsilon^{BC} = \varepsilon_{AB}\varepsilon^{BC} = -\varepsilon_{BA}\varepsilon^{CB} = \varepsilon_A^C = -\varepsilon_A^C,$$

with a canonical mapping (or isomorphism) between, e.g., k^B and k_B:

$$k^B \mapsto k_B = k^A\varepsilon_{AB}.$$



(a) Lissajous patterns representing the polarized electric field over time, viewed in the plane of incidence, resulting from the two orthogonal s and p fields, which are out of phase by the following degrees: 0, 21, 42, 64, 85, 106, 127, 148, 169 (top row); 191, 212, 233, 254, 275, 296, 319, 339, 360 (bottom row). In these Lissajous patterns, the plane polarizations are represented at 45 degrees to the axes.



(b) Representation of a polarization modulated beam over 2π in the z-direction.These are $SO(3)$ $Q^i(\omega,\delta)$ in $\boldsymbol{C}$ representations over 2π, *not* an $SU(2)$ in $Q_i(\psi,\chi)$ in $\boldsymbol{C^*}$ over π.

Fig. 2.

A potential can be defined:

$$\Phi_A = i(\varepsilon\alpha)^{-1}\nabla_A\alpha\,,$$

where α is a gauge:

$$\alpha\alpha^* = 1\,,$$

and ∇_A is a covariant derivative, $\partial/\partial x^A$, but without the commutation property. The covariant electromagnetic field is then:

$$F_{AB} = \nabla_A\Phi_B - \nabla_B\Phi_A + ig[\Phi_B,\Phi_A]\,,$$

where g is generalized charge.

A physical representation of the polarization modulated ($SU(2)$) beam can be obtained using a Lissajous pattern representation Fig. 2.

Lissajous patterns are the locus of the resultant displacement of a point which is a function of two (or more) simple periodic motions. In the usual situation, the two periodic motions are orthogonal (i.e., at right angles) and are of the same frequency. The Lissajous figures then represent the polarization of the resultant wave as a diagonal line, top left to bottom right in the case of linear perpendicular polarization; bottom left to top right, in the case of linear horizontal polarization; a series of ellipses, or a circle, in the case of circular corotating or contrarotating polarization, all of these corresponding to the possible differences in constant phase between the two simple periodic motions. If the phase is not constant, but is changing or modulated, as in the case of polarization modulation, then the pattern representing the phase is constantly changing over the time the Lissajous figure is generated. Named after Jules Lissajous (1822-1880).

We can note that the Stokes' parameters (s_0, s_1, s_2, s_3) defined over the $SU(2)$ dimensional variables, ψ, χ, of $Q_i(\psi, \chi)$ are sufficient to describe *polarization/rotation modulation*, and relate those variables to the $SO(3)$ dimensional variables, $\omega(\tau, z), \delta$, of $Q^i(\omega, \delta)$, which are sufficient to describe the static *polarization/rotation conditions* of linear, circular, left and right handed polarization/rotation.

We can also note the fundamental role that concepts of topology played in dsitinguishing *static polarization-rotation* from *polarization-rotation modulation*.

5. Aharonov-Bohm Effect

We consider now the Aharonov-Bohm effect as an example of a phenomenon understandable only from topological considerations. Beginning in 1959 Aharonov and Bohm [25] challenged the view that the classical vector potential produces no observable physical effects by proposing two experiments. The one which is most discussed is shown in Fig. 3. A beam of monoenergetic electrons exists from a source at X and is diffracted into two beams by the slits in a wall at $Y1$ and $Y2$. The two beams produce an interference pattern at III which is measured. Behind the wall is a solenoid, the $\boldsymbol{B}$ field of which points out of the paper. The absence of a free local magnetic monopole postulate in conventional $U(1)$ electromagnetism ($\nabla \bullet \boldsymbol{B} = 0$) predicts that the magnetic field outside the solenoid is zero. Before the current is turned on in the solenoid, there should be the usual interference patterns observed at III, of course, due to the differences in the two path lengths.

Aharonov and Bohm made the interesting prediction that if the current is turned on, then due to the differently directed $\boldsymbol{A}$ fields along paths 1 and 2 indicated by the arrows in Fig. 3, additional phase shifts should be discernible at III. This prediction was confirmed experimentally [26] and the evidence for the effect has been extensively reviewed [27].

It is the present writer's opinion that the topology of this situation is fundamental and dictates its explanation. Therefore we must clearly note the topology of the physical layout of the design of the situation which exhibits the effect. The physical

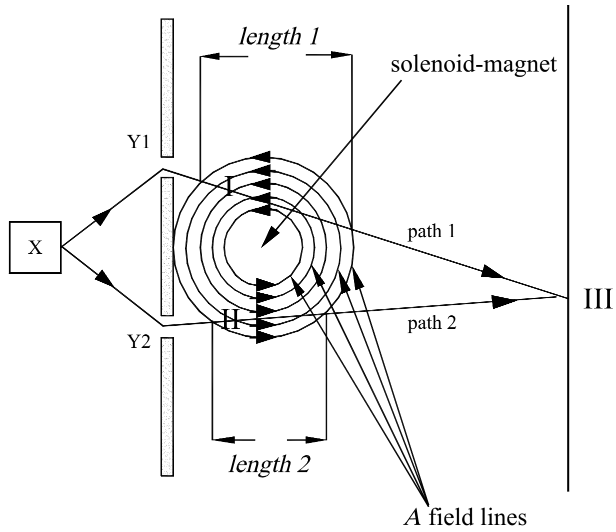


Fig. 3. Two-slit diffraction experiment of the Aharonov-Bohm effect. Electrons are produced by a source at X, pass through the slits of a mask at Y1 and Y2, interact with the $\boldsymbol{A}$ field at locations I and II over lengths l_1 and l_2, respectively, and their diffraction pattern is detected at III. The solenoid-magnet is between the slits and is directed out of the page. The different orientations of the external $\boldsymbol{A}$ field at the places of interaction I and II of the two paths 1 and 2 are indicated by arrows following the right-hand rule.

situation is that of an *interferometer*. That is, there are two paths around a central location — occupied by the solenoid — and a measurement is taken at a location, III, in the Fig 3, where there is overlap of the wave functions of the test waves which have traversed, separately, the two different paths. (The test waves or test particles are complex wave functions with phase.) It is important to note that the overlap area, at III, is the only place where a measurement can take place of the effects of the $\boldsymbol{A}$ field (which occurred earlier and at other locations (I and II). The effects of the $\boldsymbol{A}$ field occur along the two different paths and at locations I and II, but they are *inferred*, and not measurable there. Of crucial importance in this special interferometer, is the fact that the solenoid presents a *topological obstruction*. That is, if one were to consider the two joined paths of the interferometer as a raceway or a loop and one squeezed the loop tighter and tighter, then nevertheless one cannot in this situation — unlike as in most situations — reduce the interferometer's raceway of paths to a single point. (Another way of saying this is: not all closed curves in a region need have a vanishing line integral, because one exception is a loop with an obstruction.) The reason one cannot reduce the interferometer to a single point is because of the existence in its middle of the solenoid, which is a positive quantity, and acts as an obstruction.

It is the present writer's opinion that the existence of the obstruction changes the situation entirely. *Without* the existence of the solenoid in the interferometer, the loop of the two paths *can be* reduced to a single point and the region occupied by the interferometer is then *simply-connected*. But *with* the existence of the solenoid, the loop of the two paths *cannot* be reduced to a single point and the region occupied by this special interferometer is *multiply-connected*. The Aharonov-Bohm effect only exists in the multiply-connected scenario. But we should note that the Aharonov-Bohm effect is a *physical* effect and simple and multiple connectedness are *mathematical descriptions* of physical situations.

The topology of the physical interferometric situation addressed by Aharonov and Bohm defines the physics of that situation and also the mathematical description of that physics. If that situation *were not* multiply-connected, but simply-connected, then there would be no interesting physical effects to describe. The situation would be described by $U(1)$ electromagnetics and the mapping from one region to another is conventionally one-to-one. However, as the Aharonov-Bohm situation is multiply-connected, there is a two-to-one mapping $(SU(2)/Z_2)$ of the two different regions of the two paths to the single region at III where a measurement is made. Essentially, at III a measurement is made of *the differential histories* of the *two* test waves which traversed the *two* different paths and experienced two different forces resulting in two different phase effects.

In conventional, i.e., normal $U(1)$ or simply-connected situations, the fact that a vector field, viewed axially, is pointing in one direction, if penetrated from one direction on one side, and is pointing in *the opposite direction*, if penetrated from the same direction, but *on the other side*, is of no consequence at all — because that field is of $U(1)$ symmetry and can be reduced to a single point. Therefore in most cases which are of $U(1)$ symmetry, we do not need to distinguish between the direction of the vectors of a field from one region to another of that field. However, the Aharonov-Bohm situation is not conventional or simply-connected, but special. (In other words, the physical situation associated with the Aharonov-Bohm effect has a non-trivial topology). It is a multiply-connected situation and of $(SU(2)/Z_2)$ symmetry. Therefore the direction of the $\boldsymbol{A}$ field on the separate paths is of crucial importance, because a test wave traveling along one path will experience an $\boldsymbol{A}$ vectorial component directed *against* its trajectory and thus be retarded, and another test wave traveling along another path will experience an $\boldsymbol{A}$ vectorial component directed *with* its trajectory and thus its speed is boosted. These "retardations" and "boostings" can be measured as phase changes, *but not at the time nor at the locations of, I and II, where their occurrence is separated along the two different paths, but later, and at the overlap location of III.* It is important to note that if measurements are attempted at locations I and II in the Fig. 3, these effects will not be seen because there is no two-to-one mapping at either I and II and therefore no referents. The locations I and II are both simply-connected with the source and therefore only the conventional $U(1)$ electromagnetics applies at these locations (with respect to the source). It is only region III which is multiply-connected with the source and at which the histories of what happened to the test particles at I and II can be measured. In order to distinguish the "boosted" $\boldsymbol{A}$ field (because the test wave is traveling "with" its direction) from the "retarded" $\boldsymbol{A}$ field (because the test wave is traveling "against" its direction), we introduce the notation: $\boldsymbol{A}_+$ and $\boldsymbol{A}_-$.

Because of the distinction between the $\boldsymbol{A}$ oriented potential fields at positions I and II — which *are not* measurable and are *vectors or numbers* of $U(1)$ symmetry — and the $\boldsymbol{A}$ potential fields at III — which *are* measurable and are *tensors or matrix-valued functions* of (in the present instance) $(SU(2)/Z_2) = SO(3)$ symmetry (or higher symmetry) — for reasons of clarity we might introduce a distinguishing notation. In the case of the potentials of $U(1)$ symmetry at I and II we might use the lower case,$a_\mu, \mu = 0, 1, 2, 3$ and for the potentials of $(SU(2)/Z_2) = SO(3)$ at III we might use the upper case $A_\mu, \mu = 0, 1, 2, 3$. Similarly, for the electromagnetic field tensor at I and II, we might use the lower case, $\boldsymbol{f}_{\mu\nu}$, and for the electromagnetic field tensor at III, we might use the upper case, $\boldsymbol{F}_{\mu\nu}$. Then the following definitions for the electromagnetic field tensor are:
At locations I and II the Abelian relationship is:

$$\boldsymbol{f}_{\mu\nu}(x) = \partial_\nu a_\mu(x) - \partial_\mu a_\nu(x), \qquad (5)$$

where, as is well known,$\boldsymbol{f}_{\mu\nu}$ is Abelian and gauge *invariant*; But at location III the non-Abelian relationship is:

$$\boldsymbol{F}_{\mu\nu} = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) - ig_m[A_\mu(x), A_\nu(x)], \qquad (6)$$

where $\boldsymbol{F}_{\mu\nu}$ is gauge *covariant*, g_m is the magnetic charge density and the brackets are commutation brackets. We remark that in the case of non-Abelian groups, such as $SU(2)$, the potential field *can carry charge*. It is important to note that if the physical situation changes from $SU(2)$ symmetry back to $U(1)$, then $\boldsymbol{F}_{\mu\nu} \rightarrow \boldsymbol{f}_{\mu\nu}$

Despite the clarification offered by this notation, the notation can also cause confusion, because in the present literature, the electromagnetic field tensor is *always* referred to as $\boldsymbol{F}$, whether $\boldsymbol{F}$ is defined with respect to $U(1)$ or $SU(2)$ or other symmetry situations. Therefore, although we prefer this notation, we shall not proceed with it. However, it is important to note that the $\boldsymbol{A}$ field in the $U(1)$ situation is a *vector or a number*, but in the $SU(2)$ or nonAbelian situation, it is a *tensor or a matrix-valued function.*

We referred to the physical situation of the Aharonov-Bohm effect as an interferometer around an obstruction and it is 2-dimensional. It is important to note that the situation is not provided by a toroid, although a toroid is also a physical situation with an obstruction and the fields existing on a toroid are also of $SU(2)$ symmetry. However, the toroid provides a two-to-one mapping of fields in not only the x and y dimensions but also in the z dimension, and *without* the need of an electromagnetic field pointing in two directions + and −. The physical situation of the Aharonov-Bohm effect is defined only in the x and y dimensions (there is no z dimension) and in order to be of $SU(2)/Z_2$ symmetry *requires* a field to be oriented differentially on the separate paths. If the differential field is removed from the Aharonov-Bohm situation, then that situation reverts to a simple interferometric raceway which can be reduced to a single point and with no interesting physics.

How does the topology of the situation affect the explanation of an effect? A typical previous explanation [28] of the Aharonov-Bohm effect commences with the Lorentz force law:

$$\boldsymbol{F} = e\boldsymbol{E} + e\boldsymbol{v} \times \boldsymbol{B} \tag{7}$$

The electric field, $\boldsymbol{E}$, and the magnetic flux density, $\boldsymbol{B}$, are essentially confined to the inside of the solenoid and therefore cannot interact with the test electrons. An argument is developed by defining the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in terms of the $\boldsymbol{A}$ and ϕ potentials:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{8}$$

Now we can note that these conventional $U(1)$ definitions of $\boldsymbol{E}$ and $\boldsymbol{B}$ can be expanded to $SU(2)$ forms:

$$\begin{aligned} \boldsymbol{E} &= -\nabla \times \boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi, \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi. \end{aligned} \tag{9}$$

Furthermore, the $U(1)$ Lorentz force law, Eq 7, can hardly apply in this situation because the solenoid is electrically neutral to the test electrons and therefore $\boldsymbol{E} = 0$ along the two paths. Using the definition of $\boldsymbol{B}$ in Eq 9, the force law in this $SU(2)$ situation is:

$$\begin{aligned} \boldsymbol{F} &= e\boldsymbol{E} + e\boldsymbol{v} \times \boldsymbol{B} \\ &= e\left(-(\nabla \times \boldsymbol{A}) - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi\right) \\ &+ e\boldsymbol{v} \times \left(-(\nabla \times \boldsymbol{A}) - \frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi\right), \end{aligned} \tag{10}$$

but we should note that Eqs 7 and 8 are *still valid* for the conventional theory of electromagnetism based on the $U(1)$ symmetry Maxwell's equations provided in Table 1 and associated with the group $U(1)$ algebra. They are *invalid* for the theory based on the modified $SU(2)$ symmetry equations also provided in Table 1 and associated with the group $SU(2)$ algebra.

The typical explanation of the Aharonov-Bohm effect continues with the observation that a phase difference, δ, between the two test electrons is caused by the presence of the solenoid:

$$\begin{aligned} \triangle\delta &= \triangle\alpha_1 - \triangle\alpha_2 = \frac{e}{\hbar}\left(\int\limits_{l_2} A \bullet dl_2 - \int\limits_{l_1} A \bullet dl_1\right) \\ &= \frac{e}{\hbar}\int\limits_{l_2 - l_1} \nabla \times \boldsymbol{A} \bullet d\boldsymbol{S} = \frac{e}{\hbar}\int B \bullet d\boldsymbol{S} = \frac{e}{\hbar}\varphi_M\,, \end{aligned} \tag{11}$$

where $\triangle\alpha_1$ and $\triangle\alpha_2$ are the changes in the wave function for the electrons over paths 1 and 2, $\boldsymbol{S}$ is the surface area and φ_M is the *magnetic flux* defined:

$$\varphi_M = \iint A_\mu(x)dx^\mu = \iint F_{\mu\nu}d\sigma^{\mu\nu} \tag{12}$$

Now, we can extend this explanation further, by observing that the local phase change at III of the wavefunction of a test wave or particle is given by:

$$\Phi = \exp\lfloor ig_m \iint A_\mu(x)dx^\mu \rfloor = \exp[ig_m\varphi_m]\,. \tag{13}$$

Φ, which is proportional to the magnetic flux, φ_M, is known as the *phase factor* and is *gauge covariant.* Furthermore, Φ, the phase factor measured at position III is the *holonomy* of the *connection*, $\boldsymbol{A_\mu}$; and g_m is the $SU(2)$ *magnetic charge density.*

We next observe thatφ_M is in units of volt-seconds $(V.s)$ or $kg.m^2/(A.s^2) = J/A$. From Eq 11 it can be seen that $\triangle\delta$ and the phase factor, Φ, are dimensionless. Therefore we can make the prediction that if the magnetic flux, φ_M, is known and the phase factor, Φ, is measured (as in the Aharonov-Bohm situation), *the magnetic charge density, g_m, can be found by the relation*:

$$g_m = \ln(\Phi)/(i\varphi_M)\,. \tag{14}$$

Continuing the explanation: as was noted above, $\nabla \times \boldsymbol{A} = 0$ outside the solenoid and the situation must be redefined in the following way. An electron on path 1 will interact with the $\boldsymbol{A}$ field oriented in the positive direction. Conversely, an electron on path 2 will interact with the $\boldsymbol{A}$ field oriented in the negative direction. Furthermore, the $\boldsymbol{B}$ field can be defined with respect to a local stationary component $\boldsymbol{B}_1$ which is confined to the solenoid and a component $\boldsymbol{B}_2$ which is either a standing wave or propagates:

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_1 + \boldsymbol{B}_2, \\ \boldsymbol{B}_1 &= \nabla \times \boldsymbol{A}, \\ \boldsymbol{B}_2 &= -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi. \end{aligned} \tag{15}$$

The magnetic flux density, $\boldsymbol{B}_1$, is the confined component associated with $U(1) \times SU(2)$ symmetry and $\boldsymbol{B}_2$ is the propagating or standing wave component associated *only* with $SU(2)$ symmetry. In a $U(1)$ symmetry situation, $\boldsymbol{B}_1$ = components of the field associated with $U(1)$ symmetry, and $\boldsymbol{B}_2 = 0$.

The electrons traveling on paths 1 and 2 require different times to reach III from X, due to the different distances and the opposing directions of the potential $\boldsymbol{A}$ along the paths l_1 and l_2. Here we only address the effect of the opposing directions of the potential $\boldsymbol{A}$, i.e., the distances traveled are identical over the two paths. The change in the phase difference due to the presence of the $\boldsymbol{A}$ potential is then:

$$\triangle\delta = \triangle\alpha_1 - \triangle\alpha_2 = \frac{e}{\hbar}\left[\int\limits_{l_2}\left(-\frac{\partial \boldsymbol{A}_+}{\partial t} - \nabla\phi_+\right) dl_2 - \int\limits_{l_2}\left(-\frac{\partial \boldsymbol{A}_-}{\partial t} - \nabla\phi_-\right) dl_1\right] \bullet d\boldsymbol{S} = \frac{e}{\hbar}\int \boldsymbol{B}_2 \bullet d\boldsymbol{S} = \frac{e}{\hbar}\varphi_M\,. \quad (16)$$

There is no flux density $\boldsymbol{B}_1$ in this equation since this equation describes events outside the solenoid, but only the flux density $\boldsymbol{B}_2$ associated with group $SU(2)$ symmetry; and the " $+$" and " $-$" indicate the direction of the $\boldsymbol{A}$ field encountered by the test electrons – as discussed above.

We note that the phase effect is dependent on $\boldsymbol{B}_2$ and $\boldsymbol{B}_1$, but not on $\boldsymbol{B}_1$ alone. Previous treatments found no convincing argument around the fact that whereas the Aharonov-Bohm effect depends on an interaction with the $\boldsymbol{A}$ field outside the solenoid, $\boldsymbol{B}$, defined in $U(1)$ electromagnetism as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, is zero at that point of interaction. However, when $\boldsymbol{A}$ is defined in terms associated with an $SU(2)$ situation, that is not the case as we have seen.

We depart from former treatments in other ways. Commencing with a *correct* observation that the Aharonov-Bohm effect depends on the topology of the experimental situation and that the situation is not simply-connected, a former treatment then erroneously seeks an explanation of the effect in the connectedness of the $U(1)$ gauge symmetry of conventional electromagnetism, but for which (1) the potentials are ambiguously defined, (the $U(1)$ $\boldsymbol{A}$ field is gauge invariant) and (2) in $U(1)$ symmetry $\nabla \times \boldsymbol{A} = 0$ outside the solenoid.

Furthermore, whereas a former treatment again makes a *correct* observation that the non-Abelian group, $SU(2)$, is simply-connected and that the situation is governed by a multiply-connected topology, the author fails to observe that the non-Abelian group $SU(2)$ defined over the integers modulo 2, $SU(2)/Z_2$, is, in fact, multiply-connected. Because of the two paths around the solenoid it is this group which describes the topology underlying the Aharonov-Bohm effect [6]. $SU(2)/Z_2 \cong SO(3)$ is obtained from the group $SU(2)$ by identifying pairs of elements with opposite signs. The measured at location III in Fig. 3 is derived from a *single* path in $SO(3)$[2] because the *two* paths through locations I and II in $SU(2)$ are regarded as a *single* path in $SO(3)$. This path in $SU(2)/Z_2 \cong SO(3)$ cannot be shrunk to a single point by any continuous deformation and therefore adequately describes the multiple-connectedness of the Aharonov-Bohm situation. Because the former treatment failed to note the multiple connectedness of the $SU(2)/Z_2$ description of the Aharonov-Bohm situation, it *incorrectly* fell back on a $U(1)$ symmetry description.

Now back to the main point of this excursion to the Aharonov-Bohm effect: the reader will note that the author appealed to topological arguments to support the main points of his argument. Underpinning the $U(1)$ Maxwell theory is an Abelian algebra; underpinning the $SU(2)$ theory is a non-Abelian algebra. The algebras specify the form of the equations of motion. However, whether one or the other algebra can be (validly) used can only be determined by topological considerations.

6. Summary

We have attempted to show the fundamental explanatory nature of the topological description of solitons, instantons and the Aharonov-Bohm effect — and hence electromagnetism. In the case of electromagnetism we have shown elsewhere that, given a Yang-Mills description, electromagnetism can, and should be extended, in accordance with the topology with which the electromagnetic fields are associated. This approach has major implications. If the conventional theory of electromagnetism, i.e., " Maxwell's theory", which is of $U(1)$ symmetry form, is but the simplest local theory of electromagnetism, then those pursuing a unified field theory may wish to consider as a candidate for that unification, not only the simple *local* theory, but other electromagnetic fields of group symmetry higher than $U(1)$. Other such forms include symplectic gauge fields of higher group symmetry, e.g., $SU(2)$ and above.

Manuscript received November 15, 2006

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