Local Realism, Contextualism and Loopholes in Bell‘s Experiments

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Abstract

It is currently widely accepted, as a result of Bell's theorem and related experiments, that quantum mechanics is inconsistent with local realism and there is the so called quantum non-locality. We show that such a claim can be justified only in a simplified approach to quantum mechanics when one neglects the fundamental fact that there exist space and time. Mathematical definitions of local realism in the sense of Bell and in the sense of Einstein are given. We demonstrate that if we include into the quantum mechanical formalism the space-time structure in the standard way then quantum mechanics might be consistent with Einstein's local realism. It shows that loopholes are unavoidable in experiments aimed to establish a violation of Bell‘s inequalities. We show how the space-time structure can be considered from the contextual point of view. A mathematical framework for the contextual approach is outlined.

Contents

1 Introduction

1.1 Quantum non-locality without space and time?

Einstein, Podolsky and Rosen (EPR) presented an argument to show that there are situations in which the scheme of quantum theory seems to be incomplete [1]. They proposed a gedanken experiment involving a system of two particles spatially separated but correlated in position and momentum and argued that two non-commuting variables (position and momentum of a particle) can have simultaneous physical reality. They concluded that the description of physical reality given by quantum mechanics, which, due to the uncertainty principle, does not permit such a simultaneous reality, is incomplete.

Though the EPR work dealt with continuous position and momentum variables most of the further activity have concentrated almost exclusively on systems of discrete spin variables following to the Bohm [2] and Bell [3] works.

Entangled states, i.e. the states of two particles with the wave function which is not a product of the wave functions of single particles, have been studied in many theoretical and experimental works starting from works of Einstein, Podolsky and Rosen and Schrodinger.

Bell's theorem [3] states that there are quantum spin correlation functions that can not be represented as classical correlation functions of separated random variables. It has been interpreted as incompatibility of the requirement of locality with the statistical predictions of quantum mechanics [3]. For a recent discussion of Bell's theorem see, for example [4] - [18] and references therein. It is now widely accepted, as a result of Bell's theorem and related experiments, that " local realism" must be rejected and there exists the so called quantum non-locality.

However it was shown in [17, 18, 20] that in the derivation of such a conclusion the fundamental fact that space-time exists was neglected. Moreover, if we take into account the spatial dependence of the wave function then the standard formalism of quantum mechanics might be consistent with local realism.

1.2 Contextual approach

From the other side a general contextual approach to the probabilistic scheme of quantum theory was proposed in [19]. It is based on the transformation rules induced by context transitions. Context is a complex of physical conditions used for the preparation of quantum or classical states. The idea of the contextual dependence of probabilistic results of observations is a very general one. It can be used and developed in various directions. In particular it was suggested in [20] to treat boundary conditions for quantum mechanical differential equations as an appropriate context.

Context describes a measure of idealization which we use to construct a mathematical model for a physical process. For example in some approximation one can deal with models of quantum phenomena when the spatial characteristics are neglected as it was done by Bell in his consideration of the EPR paradox. However if we want to speak about fundamental properties of quantum theory then the principal role of the space-time picture should not be overlooked. In axiomatic approach to quantum theory after von Neumann [22] one often postulates only the formalism of Hilbert space, its statistical interpretation and the abstract Schrodinger evolution equation but without indication to the spatial properties of quantum system. The necessity of including into the list of basic axioms of quantum mechanics the property of covariance of the physical system under the spatial translation and rotation and moreover under the Galilei or Poincare group was stressed in [21].

In this paper we combine the spatial approach to problems of quantum non-locality from [18, 21] with the contextual approach of [19] to investigate problems of quantum non-locality and local realism. We consider the space-time as a context for a quantum model. We will present a mathematical formalism for the contextual approach. We will give also two different definitions of the notions of local realism which we call Bell's and Einstein's local realism. We demonstrate that if we include into the quantum mechanical formalism the space-time structure in the standard way then quantum mechanics actually is consistent with local realism. Since detectors of particles are obviously located somewhere in space it shows that loopholes are unavoidable in experiments aimed to establish a violation of Bell's inequalities. Our main tool will be analysis of correlation functions in quantum and in classical theory.

1.3 Bell's local realism

A mathematical formulation of Bell's local realism may be given by relation (in more details it is discussed in the next section)

$$\langle\psi|A(a)B(b)|\psi\rangle = E\xi(a)\eta(b) \tag{1}$$

Here $A(a)$ and $B(b)$ are self-adjoint operators which commute on a natural domain and a and b are certain indices. Here E is a mathematical expectation and $\xi(a)$ and $\eta(b)$ are two stochastic processes and ψ is a vector from a Hilbert space. Then we say that the triplet

$$\{A(a), B(b), \psi\}$$

satisfies the *Bell's local realism* (BLR) condition.

Bell proved that a two spin quantum correlation function which is equal to just $-a \cdot b$, where a and b are two 3-dimensional vectors, can not be represented in the form (1),

$$\langle\psi_{spin}|\sigma \cdot a \otimes \sigma \cdot b|\psi_{spin}\rangle \neq E\xi(a)\eta(b) \tag{2}$$

if one has a bound $|\xi(a)| \leq 1,\ \ |\eta(b)| \leq 1$. Here $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are two unit vectors in three-dimensional space $\mathbf{R}^3$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices,

Therefore the correlation function of two spins does not satisfy to the BLR condition (1). In this sense sometimes one speaks about quantum non-locality.

1.4 Space and time in axioms of quantum mechanics

Note however that in the previous discussion the space-time parameters were not explicitly involved though one speaks about non-locality. Actually the "local realism" in the Bell sense as it was formulated above in Eq. (1) is a notion which has nothing to do with notion of locality in the ordinary 3 dimensional space. Therefore we define also another notion which we will call the condition of local realism in the sense of Einstein.

To explain the notion let us first remind that the usual axiomatic approach to quantum theory involves only the Hilbert space, observable, the density operator ρ and the von Neumann formula for the probability $P(B)$ of the outcome B: $P(B) = Tr\rho E_B$ where $\{E_B\}$ is POVM associated with a measured space $(\Omega, \mathcal{F})$, here B belongs to the σ-algebra $\mathcal{F}$. It was stressed in [21] that in a more realistic axiomatic approach to quantum mechanics one has to includes an axiom on the existing of space and time. It can be formulated as follows

$$U(d)E_B U(d)^* = E_{\alpha_d(B)}$$

Here $U(d)$ is the unitary representation of the group of translations in time and in the three-dimensional space and $\alpha_d : \mathcal{F} \to \mathcal{F}$ is the group of automorphisms..

1.5 Einstein's local realism

Let in a Hilbert space $\mathcal{H}$ be given a family of self-adjoint operators $\{A(a, \mathcal{O})\}$ and $\{B(b, \mathcal{O})\}$ parameterized by the regions $\mathcal{O}$ in the Minkowsky space-time. Suppose that one has a representation

$$\langle\psi|A(a, \mathcal{O}_1)B(b, \mathcal{O}_2)|\psi\rangle = E\xi(a, \mathcal{O}_1)\eta(b, \mathcal{O}_2) \tag{3}$$

for $a, b, \mathcal{O}_1, \mathcal{O}_2$ for which the operators commute. Then we say that the quadruplet

$$\{A(a, \mathcal{O}_1), B(b, \mathcal{O}_2), U(d), \psi\}$$

satisfies the *Einstein local realism* (ELR) condition.

1.6 Local realist representation for quantum spin correlations

Quantum correlation describing the localized measurements of spins in the regions $\mathcal{O}_1$ and $\mathcal{O}_2$ includes the projection operators $P_{\mathcal{O}_1}$ and $P_{\mathcal{O}_2}$. In contrast to Bell's theorem (2) now there exists a local realist representation [18]

$$\langle\psi|\sigma \cdot a P_{\mathcal{O}_1} \otimes \sigma \cdot b P_{\mathcal{O}_2}|\psi\rangle = E\xi(\mathcal{O}_1, a)\eta(\mathcal{O}_2, b) \tag{4}$$

if the distance between the regions $\mathcal{O}_1$ and $\mathcal{O}_2$ is large enough. Here all classical random variables are bounded by 1.

Since detectors of particles are obviously located somewhere in space it shows that loopholes are unavoidable in experiments aimed to establish a violation of Bell's inequalities. Though there were some reports on experimental derivation of violation of Bell's inequalities, in fact such violations always were based on additional assumptions besides local realism. No genuine Bell's inequalities have been violated since always some loopholes were in the experiments, for a review see for example [4, 13]. There were many discussions of proposals for experiments which could avoid the loopholes however up to now a convincing proposal still did not advanced .

One can compare the situation with attempts to measure the position and momentum of a particle in a single experiment. Also one could speak about some technical difficulties (similar to the efficiency of detectors loophole) and hope that some could come with a proposal to make an experiment without loopholes. However we know from the uncertainty relation for the measurement of momentum and position that it is not possible. Similarly the formula (4) shows that a loophole free experiment in which a violation of Bell's inequalities will be observed is impossible if the distance between detectors is large enough. Therefore loopholes in Bell's experiments are irreducible.

1.7 EPR versus Bohm and Bell

The original EPR system involving continuous variables has been considered by Bell in [29]. He has mentioned that if one admits "measurement" of arbitrary "observable" at arbitrary state than it is easy to mimic his work on spin variables (just take a two-dimensional subspace and define an analogue of spin operators). The problem which he was discussing in [29] is narrower problem, restricted to measurement of positions only, on two non-interacting spin-less particles in free space. Bell used the Wigner distribution approach to quantum mechanics.The original EPR state has a nonnegative Wigner distribution. Bell argues that it gives a local, classical model of hidden variables and therefore the EPR state should not violate local realism. He then considers a state with non-positive Wigner distribution and demonstrates that this state violates local realism.

Bell's proof of violation of local realism in phase space has been criticized in [30] because of the use of an unnormalizable Wigner distribution. Then in [31] it was demonstrated that the Wigner function of the EPR state, though positive definite, provides an evidence of the nonlocal character of this state if one measures a parity operator.

In [32] we have applied to the original EPR problem the method which was used by Bell in his well known paper [3]. He has shown that the correlation function of two spins cannot be represented by classical correlations of separated bounded random variables. This Bell‘s theorem has been interpreted as incompatibility of local realism with quantum mechanics. It was shown in [32] that, in contrast to Bell‘s theorem for spin correlation functions, the correlation function of positions (or momenta) of two particles always admits a representation in the form of classical correlation of separated random variables.The following representation was proved

$$\langle\psi|q_1(\alpha_1)q_2(\alpha_2)|\psi\rangle = E\xi_1(\alpha_1)\xi_2(\alpha_2) \tag{5}$$

The explanation of the notations see below. Therefore we obtain a local realistic (in the sense of Bell and in the sense of Einstein as well) representation for the correlation function in the original EPR model. This result looks rather surprising since one thinks that the Bohm-Bell reformulation of the EPR paradox is equivalent to the original one.

2 Correlation functions and local realism

A mathematical formulation of Bell's local realism may be given as follows. Let, in a Hilbert space $\mathcal{H}$, be given two families of self-adjoint operators $\{A(a)\}$ and $\{B(b)\}$ which commute $[A(a), B(b)] = 0$ on a natural domain. Here a and b are elements of two arbitrary sets of indices. Suppose that one has a representation

$$\langle\psi|A(a)B(b)|\psi\rangle = E\xi(a)\eta(b) \tag{6}$$

for any a, b where E is a mathematical expectation and $\xi(a)$ and $\eta(b)$ are two stochastic processes such that the range of $\xi(a)$ is the spectrum of $A(a)$ and the range of $\eta(b)$ is the spectrum of $B(b)$. Here ψ is a vector from $\mathcal{H}$. Then we say that *the triplet*

$$\{\{A(a)\}, \{B(b)\}, \psi\}$$

satisfies the BLR (Bell‘s local realism) condition.

Bell proved that a two spin quantum correlation function which is equal to just $-a \cdot b$, where a and b are two 3-dimensional vectors, can not be represented in the form (13) if one has a bound $|\xi(a)| \leq 1,\ \ |\eta(b)| \leq 1$. Therefore the correlation function of two spins does not satisfy to the BLR condition (6). In this sense sometimes one speaks about quantum non-locality.

Note however that in the previous discussion the space-time parameters were not explicitly involved though one speaks about non-locality. Actually the "local realism" in the Bell sense as it was formulated above in Eq. (13) is a very general notion which has nothing to do with notion of locality in the ordinary three-dimensional space. We will define now another notion which we will call the condition of local realism in the sense of Einstein. First let us recall that in quantum field theory the condition of locality (local commutativity) reads:

$$[F(x), G(y)] = 0 \tag{7}$$

if the space-time points x and y are space-like separated. Here $F(x)$ and $G(y)$ are two Bose field operators (for Fermi fields we have anti-commutator).

Let in the Hilbert space $\mathcal{H}$ there is a unitary representation U of the inhomogeneous Lorentz group and let be given a family of self-adjoint operators $\{A(a, \mathcal{O})\}$ parameterized by the regions $\mathcal{O}$ in Minkowsky space-time where a is an arbitrary index. Let us suppose that the unitary operator translations act as

$$U(d)A(a, \mathcal{O})U(d)^* = A(a, \mathcal{O}(d)) \tag{8}$$

where d is a four dimensional vector and $\mathcal{O}(d))$ is a shift of $\mathcal{O}$ at d. Let be given also a family of operators $\{B(b, \mathcal{O})\}$ with similar properties. Suppose that one has a representation

$$\langle\psi|A(a, \mathcal{O}_1)B(b, \mathcal{O}_2)|\psi\rangle = E\xi(a, \mathcal{O}_1)\eta(b, \mathcal{O}_2) \tag{9}$$

for $a, b, \mathcal{O}_1, \mathcal{O}_2$ for which the operators commute

$$[A(a, \mathcal{O}_1), B(b, \mathcal{O}_2)] = 0$$

The correlation function (9) describes the results of a simultaneous measurement. Moreover we suppose that the range of $\xi(a, \mathcal{O}_1)$ is the spectrum of $A(a, \mathcal{O}_1)$ and the range of $\eta(b, \mathcal{O}_2)$ is the spectrum of $B(b, \mathcal{O}_2)$. Then we say that *the quadruplet*

$$\{\{A(a, \mathcal{O}_1)\}, \{B(b, \mathcal{O}_2)\}, U, \psi\}$$

satisfies the ELR (Einstein local realism) condition.

For Fermi fields which anti-commute we assume the same relation (9) but the random fields ξ and η should be now anti-commutative random fields (superanalysis and probability with anticommutative variables are considered in [33, 34]).

One can write an analogue of the presented notions in the case when the region $\mathcal{O}$ shrinks to a point (in such a case we have an operator $A(a, x)$) and also for n-point correlation functions

$$\langle\psi|A_1(a_1, x_1)...A_n(a_n, x_n)|\psi\rangle = E\xi_1(a_1, x_1)...\xi_n(a_n, x_n) \tag{10}$$

A non-commutative spectral theory related with such representations is considered in [20].

3 Contextual classical and quantum probability

The contextual probabilistic approach is nothing than probabilistic formalization of Bohr's idea that the whole experimental arrangement must be taken into account. The basic postulate of the contextual probabilistic approach to general statistical measurements is that **probability distributions for physical variables depend on complexes of experimental physical conditions.** Such complexes are called (experimental) contexts. Mathematically contextualism means the impossibility to operate with an abstract (e.g. Kolmogorov) probability $\mathbf{P}$ indending of a context. Thus, in the opposite to traditions in probability theory, we could not work with e.g. a single Kolmogorov probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that was fixed once and for ever. If we choose the measure theoretical approach to probability (Kolmogorov, 1933), then in the contextualist probabilistic framework we should to work with families of Kolmogorov probability spaces. Here mathematically every context is represented by its own probability space, compare to the camelion approach of L. Accardi [35] and the theory of probability manifolds of S. Gudder [36] .

Moreover, Kolmogorov's measure-theoretical probability theory [37] is not so natural as the mathematical base for the contextual probabilistic approach to statistical measurements. The main contribution of A. N. Kolmogorov into axiomatisation of probability theory was consideration of abstract probability measures. The great advantage of the Kolmogorov probability theory was the possibility to perform general probabilistic derivations, i.e., derivations for abstract probabilities without to take into account contextual dependence of probabilities.

But in the contextual probabilistic framework it would be more natural to start not with an abstract probability, but directly with a context and then consider a sequence of experimental trails in this context. As the result, we get a sequence of physical characteristics of systems under consideration. Then we can define the probability distribution (if it exists at all) of those characteristics by using the *principle of statistical stabilization* of relative frequencies. Mathematical formalization of this approach (*frequency probability theory*) was proposed by R. von Mises [38] on the basis of theory of *collectives* (random sequences). Thus if we use the frequency probability theory, then we can identify a context with a collective. Our fundamental thesis *"first context – then probability distribution"* is closely related to von Mises' fundamental thesis: *"first collective – then probability distribution".*

The authors of the paper are well aware that the original von Mises definition of collective was not mathematically rigorous, see e.g. on the details. This problem induced extended investigations on the notion of randomness, see e.g. [39], [40]. In particular, those investigations induced the theory of recursive functions and Kolmogorov's algorithmic complexity. We recall that, in particular, if we restrict the class of von Mises place selections to recursive functions, then we get mathematically well defined theory of collectives. However, the present paper is far away all those problems with the notion of randomness. All our considerations are related only to the statistical stabilization of relative frequencies. We do not take care on randomness. We belief that all sequence induced by e.g. quantum statistical experiments are random.

Mathematical formalization of the notion of context in general case is a problem of large complexity. In this paper we propose the following definition of *quantum conmtext.*

Definition. Every family $\mathcal{A} = \{A_1, A_2, ...\}$ (finite or infinite) of self-adjoint commutative operators is said to be a quantum context.

Example 1. (Space-time context). Let $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ be the system of generators of the unitary group of translations. Then $\mathcal{A}$ is said to be the space-time context.

Example 2. (Internal symmetry). Let G by a compact Lie group if internal symmetries (for example, the gauge group $U(1)$ which describes the electric charge). Then generators of the unitary representation of the group defines the internal symmetry context.

4 Bell's Theorem and Stochastic Processes

In the presentation of Bell's theorem we will follow [17] where one can find also more references. *Bell's theorem,* as it is formulated in [17], reads:

$$(a, b) \neq E\xi(a)\eta(b) \tag{11}$$

were $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ are two unit vectors in three-dimensional space $\mathbf{R}^3$. Here $\xi(a)=\xi(a,\lambda)$ and $\eta(b)=\eta(b,\lambda)$ are random fields on the sphere, λ is an element from the probability space $(\Lambda,\mathcal{F},d\rho(\lambda))$. Here Λ is a set, $\mathcal{F}$ is a sigma-algebra of subsets and $d\rho(\lambda)$ is a probability measure, i.e. $d\rho(\lambda)\geq 0,\ \int d\rho(\lambda)=1$. The expectation is

$$Ef=\int_\Lambda f(\lambda)d\rho(\lambda)$$

The random fields satisfy the bound

$$|\xi(a,\lambda)|\leq 1,\ \ |\eta(b,\lambda)|\leq 1 \tag{12}$$

The theorem says that there exists no probability space and a pair of stochastic processes with indicated properties such that their expectation is equal to the scalar product of the vectors a and b. The form (11) is convenient for various generalizations. o

Let us discuss now a physical interpretation of this result. In the Bohm formulation of the EPR argument one considers a pair of spin one-half particles formed in the singlet spin state and moving freely towards two detectors. If one neglects the space part of the wave function then one has the Hilbert space $C^2\otimes C^2$ and the quantum mechanical correlation of two spins in the singlet state $\psi_{spin}\in C^2\otimes C^2$ is

$$D_{spin}(a,b)=\langle\psi_{spin}|\sigma\cdot a\otimes\sigma\cdot b|\psi_{spin}\rangle=-a\cdot b \tag{13}$$

Here $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ are two unit vectors in three-dimensional space $\mathbf{R}^3$, $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices, $\sigma\cdot a=\sum_{i=1}^3\sigma_i a_i$ and

$$\psi_{spin}=\frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right).$$

The proof of the theorem is based on Bell's or the Clauser-Horn-Shimony-Holt (CHSH) inequalities. Let us stress that the main point in the mathematical proof is actually not the discretness of the classical or quantum spin variables and even not a nonlocality but the bound (12) for classical random fields.

4.1 Classical model of spin correlation

To explain the last point we present here a simple local in the sense of Bell classical probabilistic model which reproduces the quantum mechanical correlation of two spins. Let us take as a probability space Λ just 3 points: $\Lambda=\{1,2,3\}$ and the expectation

$$Ef=\frac{1}{3}\sum_{\lambda=1}^{3}f(\lambda)$$

Let the random fields be

$$\xi(a,\lambda)=\eta(a,\lambda)=\sqrt{3}a_\lambda,\quad \lambda=1,2,3$$

Then one has the relation:

$$(a,b)=E\xi(a)\xi(b)$$

The Bell‘s theorem (11) does not valid in this case because we do not have the bound (12). Instead we have

$$|\xi(a,\lambda)| \leq \sqrt{3}$$

This model shows that the bound (12) plays the crucial role in the proof of Bell‘s theorem. Actually to reproduce (a,b) we can use even a deterministic model: simply the first experimentalist will report about the measurement of the components of the vector (a_1,a_2,a_3) and the second about the measurement of the components of the vector (b_1,b_2,b_3).

4.2 CHSH Inequality

The proof of Bell's theorem is based on the following theorem which is a slightly generalized the Clauser-Horn-Shimony-Holt (CHSH) result.

Theorem 2. Let $f_1,\ f_2,\ g_1$ and g_2 be random variables (i.e. measured functions) on the probability space $(\Lambda, \mathcal{F}, d\rho(\lambda))$ such that

$$|f_i(\lambda)g_j(\lambda)| \leq 1, \quad i,j=1,2. \tag{14}$$

Denote

$$P_{ij} = Ef_ig_j, \quad i,j=1,2.$$

Then

$$|P_{11}-P_{12}| + |P_{21}+P_{22}| \leq 2 \tag{15}$$

The last inequality is called the CHSH inequality.

5 Correlation functions in EPR model

Now let us apply similar approach to the original EPR case [32]. The Hilbert space of two one-dimensional particles is $L^2(R)\otimes L^2(R)$ and canonical coordinates and momenta are q_1,q_2,p_1,p_2 which obey the commutation relations

$$[q_m,p_n]=i\delta_{mn}, \quad [q_m,q_n]=0, \quad [p_m,p_n]=0, \quad m,n=1,2 \tag{16}$$

The EPR paradox can be described as follows. There is such a state of two particles that by measuring p_1 or q_1 of the first particle, we can predict with certainty and without interacting with the second particle, either the value of p_2 or the value of q_2 of the second particle. In the first case p_2 is an element of physical reality, in the second q_2 is. Then, these realities must exist in the second particle before any measurement on the first particle since it is assumed that the particle are separated by a space-like interval. However the realities can not be described by quantum mechanics because they are incompatible – coordinate and momenta do not commute. So that EPR conclude that quantum mechanics is not complete. Note that the EPR state actually is not a normalized state since it is represented by the delta-function, $\psi=\delta(x_1-x_2-a)$.

An important point in the EPR consideration is that one can choose what we measure – either the value of p_1 or the value of q_1.

For a mathematical formulation of a free choice we introduce canonical transformations of our variables:

$$q_n(\alpha) = q_n \cos\alpha - p_n \sin\alpha, \quad p_n(\alpha) = q_n \sin\alpha + p_n \cos\alpha; \quad n = 1,2 \tag{17}$$

Then one gets

$$[q_m(\alpha), p_n(\alpha)] = i\delta_{mn}; \quad n = 1,2 \tag{18}$$

In particular one has $q_n(0) = q_n, \quad q_n(3\pi/2) = p_n, \quad n = 1,2.$

Now let us consider the correlation function

$$D(\alpha_1, \alpha_2) = \langle\psi|q_1(\alpha_1) \otimes q_2(\alpha_2)|\psi\rangle \tag{19}$$

The correlation function $D(\alpha_1, \alpha_2)$ (19) is an analogue of the Bell correlation function $D_{spin}(a, b)$ (13). Bell in [29] has suggested to consider the correlation function of just the free evolutions of the particles at different times (see below).

We are interested in the question whether the quantum mechanical correlation function (19) can be represented in the form

$$\langle\psi|q_1(\alpha_1) \otimes q_2(\alpha_2)|\psi\rangle = E\xi_1(\alpha_1)\xi_2(\alpha_2) \tag{20}$$

Here $\xi_n(\alpha_n) = \xi_n(\alpha_n, \lambda), n = 1,2$ are two real random processes, possibly unbounded The parameters λ are interpreted as hidden variables in a realist theory.

Theorem. For an arbitrary state $\psi \in L^2(R) \otimes L^2(R)$ on which products of operators q_1, q_2, p_1, p_2 are defined there exist random processes $\xi_n(\alpha_n, \lambda)$ such that the relation (20) is valid.

Proof. We rewrite the correlation function $D(\alpha_1, \alpha_2)$ (19) in the form

$$\langle\psi|q_1(\alpha_1) \otimes q_2(\alpha_2)|\psi\rangle =< q_1q_2 > \cos\alpha_1 \cos\alpha_2 - < p_1q_2 > \sin\alpha_1 \cos\alpha_2 \tag{21}$$

$$- < q_1p_2 > \cos\alpha_1 \sin\alpha_2 + < p_1p_2 > \sin\alpha_1 \sin\alpha_2$$

Here we use the notations as

$$< q_1q_2 >= \langle\psi|q_1q_2|\psi\rangle$$

Now let us set

$$\xi_1(\alpha_1, \lambda) = f_1(\lambda) \cos\alpha_1 - g_1(\lambda) \sin\alpha_1,$$

$$\xi_2(\alpha_2, \lambda) = f_2(\lambda) \cos\alpha_2 - g_2(\lambda) \sin\alpha_2$$

Here real functions $f_n(\lambda), g_n(\lambda), \ n = 1,2$ are such that

$$Ef_1f_2 =< q_1q_2 >, \ Eg_1f_2 =< p_1q_2 >, \ Ef_1g_2 =< q_1p_2 >, \ Eg_1g_2 =< p_1p_2 > \tag{22}$$

We use for the expectation the notations as $Ef_1f_2 = \int f_1(\lambda)f_2,(\lambda)d\rho(\lambda)$. To solve the system of equations (22) we take

$$f_n(\lambda) = \sum_{\mu=1}^{2} F_{n\mu}\eta_\mu(\lambda), \ g_n(\lambda) = \sum_{\mu=1}^{2} G_{n\mu}\eta_\mu(\lambda) \tag{23}$$

where $F_{n\mu}, G_{n\mu}$ are constants and $E\eta_\mu\eta_\nu = \delta_{\mu\nu}$. We denote

$$< q_1q_2 >= A, \quad < p_1q_2 >= B, \quad < q_1p_2 >= C, \quad < p_1p_2 >= D.$$

A solution of Eqs (22) may be given for example by

$$f_1 = A\eta_1, \quad f_2 = \eta_1,$$

$$g_1 = B\eta_1 + (D - \frac{BC}{A})\eta_2, \; g_2 = \frac{C}{A}\eta_1 + \eta_2$$

Hence the representation of the quantum correlation function in terms of the separated classical random processes (20) is proved.

Remark 1. We were able to solve the system of equations (22) because there are no bounds to the random variables f_1, f_2, g_1, g_2. In the case of the Bohm spin model one has the bound (14) which leads to the CSHS inequality (15) and as a result an analogue of equations (22) in the Bohm model has no solution.

Remark 2. The condition of reality of the functions $\xi_n(\alpha_n, \lambda)$ is important. It means that the range of $\xi_n(\alpha_n, \lambda)$ is the set of eigenvalues of the operator $q_n(\alpha_n)$. If we relax this condition then one can get a hidden variable representation just by using an expansion of unity:

$$\langle\psi|q_1(\alpha_1)q_2(\alpha_2)|\psi\rangle = \sum_\lambda \langle\psi|q_1(\alpha_1)|\lambda\rangle \langle\lambda|q_2(\alpha_2)|\psi\rangle$$

For a discussion of this point in the context of a noncommutative spectral theory see [18].

Similarly one can prove a representation

$$\langle\psi|q_1(t_1) \otimes q_2(t_2)|\psi\rangle = \int \xi_1(t_1, \lambda)\xi_2(t_2, \lambda)d\rho(\lambda) \tag{24}$$

where $q_n(t) = q_n + p_n t, \; n = 1, 2$ is a free quantum evolution of the particles. It is enough to take

$$\xi_1(t_1, \lambda) = f_1(\lambda) + g_1(\lambda)t_1, \quad \xi_2(t_2, \lambda) = f_2(\lambda) + g_2(\lambda)t_2.$$

Remark 3. In fact we can prove a more general theorem. If $f(s, t)$ is a function of two variables then it can be represented as the expectation of two stochastic processes: $f(s, t) = E\xi(s)\eta(t)$. Indeed, if $f(s, t) = \sum_n g_n(s)h_n(t)$ then we can take

$$\xi(s, \omega) = \sum_n g_n(s)x_n(\omega), \; \eta(t, \omega) = \sum_n h_n(s)x_n(\omega)$$

where $Ex_nx_m = \delta_{nm}$.

6 Space-time dependence of correlation functions and disentanglement

6.1 Modified Bell's equation

In the previous sections the space part of the wave function of the particles was neglected. However exactly the space part is relevant to the discussion of locality. The Hilbert space assigned to one particle with spin 1/2 is $\mathbf{C}^2 \otimes L^2(\mathbf{R}^3)$ and the Hilbert space of two particles is $\mathbf{C}^2 \otimes L^2(\mathbf{R}^3) \otimes \mathbf{C}^2 \otimes L^2(\mathbf{R}^3)$. The complete wave function is $\psi = (\psi_{ij}(\mathbf{r}_1, \mathbf{r}_2, t))$ where i and j are spinor indices, t is time and $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors in three-dimensional space.

We suppose that there are two detectors (A and B) which are located in space $\mathbf{R}^3$ within the two localized regions $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, well separated from one another. If one makes a local observation in the region $\mathcal{O}_1$ then this means that one measures not only the spin observable σ_i but also some another observable which describes the localization of the particle like the energy density or the projection operator $P_{\mathcal{O}}$ to the region $\mathcal{O}$. Normally in experiments there are polarizers and detectors. We will consider here correlation functions which includes the projection operators $P_{\mathcal{O}}$.

Quantum correlation describing the localized measurements of spins in the regions $\mathcal{O}_1$ and $\mathcal{O}_2$ is

$$\omega(\sigma \cdot aP_{\mathcal{O}_1} \otimes \sigma \cdot bP_{\mathcal{O}_2}) = \langle \psi | \sigma \cdot aP_{\mathcal{O}_1} \otimes \sigma \cdot bP_{\mathcal{O}_2} | \psi \rangle \tag{25}$$

Let us consider the simplest case when the wave function has the form of the product of the spin function and the spacial function $\psi = \psi_{spin}\phi(\mathbf{r}_1, \mathbf{r}_2)$. Here $\phi(\mathbf{r}_1, \mathbf{r}_2)$ is a complex valued function. Then one has

$$\omega(\sigma \cdot aP_{\mathcal{O}_1} \otimes \sigma \cdot bP_{\mathcal{O}_2}) == g(\mathcal{O}_1, \mathcal{O}_2) D_{spin}(a, b) \tag{26}$$

where the function

$$g(\mathcal{O}_1, \mathcal{O}_2) = \int_{\mathcal{O}_1 \times \mathcal{O}_2} |\phi(\mathbf{r}_1, \mathbf{r}_2)|^2 d\mathbf{r}_1 d\mathbf{r}_2 \tag{27}$$

describes correlation of particles in space. It is the probability to find one particle in the region $\mathcal{O}_1$ and another particle in the region $\mathcal{O}_2$.

One has

$$0 \leq g(\mathcal{O}_1, \mathcal{O}_2) \leq 1 \tag{28}$$

6.2 Disentanglement

If $\mathcal{O}_1$ is a bounded region and $\mathcal{O}_1(l)$ is a translation of $\mathcal{O}_1$ to the 3-vector l then one can prove

$$\lim_{|l| \to \infty} g(\mathcal{O}_1(l), \mathcal{O}_2) = 0 \tag{29}$$

Since

$$\langle \psi_{spin} | \sigma \cdot a \otimes I | \psi_{spin} \rangle = 0$$

we have

$$\omega(\sigma \cdot aP_{\mathcal{O}_1} \otimes I) = 0.$$

Therefore we have proved the following proposition which says that the state $\psi = \psi_{spin}\phi(\mathbf{r}_1, \mathbf{r}_2)$ becomes disentangled (factorized) at large distances.

Proposition. One has the following property of the asymptotic factorization (disentanglement) at large distances:

$$\lim_{|l| \to \infty} [\omega(\sigma \cdot aP_{\mathcal{O}_1(l)} \otimes \sigma \cdot bP_{\mathcal{O}_2}) - \omega(\sigma \cdot aP_{\mathcal{O}_1(l)} \otimes I)\omega(I \otimes \sigma \cdot bP_{\mathcal{O}_2})] = 0 \tag{30}$$

or

$$\lim_{|l|\to\infty} \omega(\sigma \cdot a P_{\mathcal{O}_1(l)} \otimes \sigma \cdot b P_{\mathcal{O}_2}) = 0.$$

Now one inquires whether one can write a representation

$$\omega(\sigma \cdot a P_{\mathcal{O}_1} \otimes \sigma \cdot b P_{\mathcal{O}_2}) = \int \xi_1(a, \mathcal{O}_1, \lambda) \xi_2(b, \mathcal{O}_2, \lambda) d\rho(\lambda) \quad (31)$$

where $|\xi_1(a, \mathcal{O}_1, \lambda)| \leq 1, \quad |\xi_2(b, \mathcal{O}_2, \lambda)| \leq 1$.

Remark. A local modified equation reads

$$|\phi(\mathbf{r_1}, \mathbf{r_2}, t)|^2(a, b) = E\xi(a, \mathbf{r_1}, t)\eta(b, \mathbf{r_2}, t).$$

If we are interested in the conditional probability of finding the projection of spin along vector a for the particle 1 in the region $\mathcal{O}_1$ and the projection of spin along the vector b for the particle 2 in the region $\mathcal{O}_2$ then we have to divide both sides of Eq. (31) by $g(\mathcal{O}_1, \mathcal{O}_2)$.

Note that here the classical random variable $\xi_1 = \xi_1(a, \mathcal{O}_1, \lambda)$ is not only separated in the sense of Bell (i.e. it depends only on a) but it is also local in the 3 dim space since it depends only on the region $\mathcal{O}_1$. The classical random variable ξ_2 is also local in 3 dim space since it depends only on $\mathcal{O}_2$. Note also that since the eigenvalues of the projector $P_{\mathcal{O}}$ are 0 or 1 then one should have $|\xi_n(a, \mathcal{O}_n)| \leq 1, \ n = 1, 2$.

Due to the property of the asymptotic factorization and the vanishing of the quantum correlation for large $|l|$ there exists a trivial asymptotic classical representation of the form (31) with $\xi = \eta = 0$.

We can do even better and find a classical representation which will be valid uniformly for large $|l|$.

Let us take now the wave function ϕ of the form $\phi = \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)$ where

$$\int_{R^3} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 = 1, \quad \int_{R^3} |\psi_2(\mathbf{r}_2)|^2 d\mathbf{r}_2 = 1$$

In this case

$$g(\mathcal{O}_1(l), \mathcal{O}_2) = \int_{\mathcal{O}_1(l)} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 \cdot \int_{\mathcal{O}_2} |\psi_2(\mathbf{r}_2)|^2 d\mathbf{r}_2$$

There exists such $L > 0$ that

$$\int_{B_L} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 = \epsilon < 1/2,$$

where $B_L = \{\mathbf{r} \in R^3 : |\mathbf{r}| \geq L\}$.

We have the following

Theorem 4. Under the above assumptions and for large enough $|l|$ there exists the following representation of the quantum correlation function

$$\omega(\sigma \cdot a P_{\mathcal{O}_1(l)} \otimes \sigma \cdot b P_{\mathcal{O}_2}) = E\xi(\mathcal{O}_1(l), a)\xi(\mathcal{O}_2, b)$$

where all classical random variables are bounded by 1.

7 Conclusions

Mathematical definitions of local realism in the sense of Bell and in the sense of Einstein are given in the paper. We show how the space-time structure can be considered from the contextual point of view. A mathematical framework for the contextual approach is outlined. We demonstrate that if we include into the quantum mechanical formalism the space-time structure in the standard way then quantum mechanics might be consistent with Einstein's local realism. It shows that loopholes are unavoidable in experiments aimed to establish a violation of Bell‘s inequalities.

It is shown also that, in contrast to the Bell‘s theorem for the spin or polarization variables, for the original EPR correlation functions which deal with positions and momenta one can get a local realistic representation in terms of separated random processes. The representation is obtained for any state including entangled states. Therefore the original EPR model does not lead to quantum nonlocality in the sense of Bell even for entangled states. One can get quantum nonlocality in the EPR situation only if we rather artificially restrict ourself in the measurements with a two dimensional subspace of the infinite dimensional Hilbert space corresponding to the position or momentum observables. An interrelation of the roles of entangled states and the bounded observables in considerations of local realism and quantum nonlocality deserves a further study.

It is important to develop further the mathematical theory of context in classical and in quantum theory.

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