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AN IMPROVED SEARCH-EXTENTION METHOD FOR SOLVING SEMILINEAR PDES*

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Abstract This article will combine the finite element method, the interpolated coefficient finite element method, the eigenfunction expansion method, and the search-extension method to obtain the multiple solutions for semilinear elliptic equations. This strategy not only grently reduces the expensive computation, but also is successfully implemented to obtain multiple solutions for a class of semilinear elliptic boundary value problems with non-odd nonlinearity on some convex or nonconvex domains. Numerical solutions illustrated by their graphics for visualization will show the efficiency of the approach.

Key words Semilinear PDES, interpolated coefficient finite element method, multiple solutions, improved search-extension

2000 MR Subject Classification 65N30

1 Introduction

Consider the semilinear elliptic Dirichlet BVP:

$$\triangle u + f(x,u) = 0 \ \text{ in } \ \Omega, \ u = 0 \ \text{ on } \ \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in R^d with a regular boundary $\partial\Omega$ and the function $f(x,t)$ satisfies the following hypotheses:

(A1) $f(x,t) \in C^1(\overline{\Omega} \times R, R)$;

(A2) there are constants C_1 and C_2 such that

$$|f(x,t)| \le C_1 + C_2|t|^p,$$

where $0 \le p < \frac{d+2}{d-2}$ for $d \ge 3$. If $d = 2$,

$$|f(x,t)| \le C_3 \exp(\psi(t)),$$

where $\psi(t)/t^2 \to 0$ as $t \to \infty$ and C_3 is a constant.

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(A3) $f(x,0) = f'_t(x,0) = 0$;

(A4) there are constants $\mu > 2$ and $M > 0$, such that for $|t| \geq M$, $0 < \mu F(x,t) \leq tf(x,t)$. It is well-known that the functional $J : H_0^1(\Omega) \to R$ defined by

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(x,u)\right) \mathrm{d}x, \tag{1.2}$$

where $F(x,u) = \int_0^u f(x,t)\mathrm{d}t$, is C^1 and satisfies the PS condition (see [1], [2]). 0 is a local minimum point of J, whose Morse index $MI = 0$. It is easy to verify that the critical points of $J(u)$ correspond to weak solutions of (1.1). It is known [3] that (1.1) has an infinite number of solutions if $f(x,t)$ is of odd nonlinearity and $F(x,t) > 0$ for $|t| \geq M$. Without assuming the oddness of $f(x,t)$ w.r.t. t, the study of the multiple solutions of (1.1) becomes much more challenging. The sharpest result so far was obtained by Z.Q.Wang [4], who used linking and Morse type arguments to verify that (1.1) has at least three nontrivial solutions, two of which are the positive and negative mountain pass solutions under the hypotheses (A1) to (A4). In more recent articles by Castro et al. [5] and Bartsch and Wang [6], it was further proved that the third nontrivial solution claimed in [4] was a sign-changing solution with Morse index 2.

In recent years, several constructive methods, e.g., the Mountain Pass Algorithm (MPA), the High Linking Algorithm (HLA), the Minimax Algorithm (MNA), and the Search-extension Method (SEM) have been developed (see [7], [8], [9] and [10] resp.). However, in general, the MPA can only get the two positive and negative mountain pass solutions with Morse index 1 or 0. Due to the fact that the HLA depends on some local behaviors of the known critical points, the HLA can find at most four solutions of (1.1), two of which are sign-changing solutions with $MI = 2$, even if $f(x,t)$ is of odd nonlinearity. The MNA is dependent on the continuity of the peak selection, the separation condition, and an ascent direction of the known critical point and thus can only obtain the critical points with "nice" properties. When $f(x,t)$ is of odd nonlinearity and the domain is symmetric, the SEM can produce an infinite number of solutions of (1.1), whose structure and distribution are connected with the eigenvalues of $-\triangle$. Nevertheless, the fact that $f(x,t)$ is of general nonlinearity and the domain is nonsymmetric makes the computation very large and expensive.

In this article, we suggest an improved search-extension method (ISEM) which combines the finite element method, the interpolated coefficient finite element method, and the SEM to compute the solutions of (1.1) with non-odd nonlinearity on some symmetric and nonsymmetric domains. One will see that our approach is very robust numerically.

2 The Interpolated Coefficient Finite Element Method

As the interpolated coefficient finite element method plays a crucial role in the ISEM, we first introduce some background and results about it.

Denote a subspace $S_0 = \{u \in H^1(\Omega), u = 0 \text{ on } \partial\Omega\}$. Consider the weak form of a semilinear elliptic problem with zero Dirichlet boundary condition, that is,

$$Q(u,v) = A(u,v) - (f(u),v) - (g(x),v) = 0,\ v \in S_0, \tag{2.1}$$

where Ω is a d-dimensional bounded domain with boundary $\partial\Omega$ and the bilinear form

$$A(u,v) = \int_\Omega (a_{ij}(x)D_i u D_j v + a(x)uv)\mathrm{d}x \tag{2.2}$$

is assumed to be bounded and S_0-coercive. Furthermore, it is supposed that $d \leq 3$ and the problem (2.1) has a solitary solution u.

For the sake of the implementation of the finite element method and the interpolated coefficient finite method in the subsequent discussions, the domain Ω is subdivided into a finite number of elements τ with the subdivision J^h quasi-uniform and $Z_h = \{x_j\}_1^M$, the set of all interior nodes. Denote $S^h \subset S_0$ the n-degree finite element subspace and $\{N_j(x)\}_1^M$ the base of S^h. It is well known that the classical finite element solution $u_h \in S^h$ of (2.1) can be expressed as $u_h(x) = \sum\limits_{j=1}^{M} U_j N_j(x)$ with $U_j = u_h(x_j)$ and satisfies

$$A(u_h, v) - (f(u_h), v) = (g, v), v \in S^h. \tag{2.3}$$

By taking $v = N_i$, $i = 1, 2, \cdots, M$, (2.3) leads to a nonlinear algebraic system of equations

$$\sum_{j=1}^{M} A(N_j, N_i)U_j - (f(\sum_{j=1}^{M} N_j(x)U_j), N_i) = (g, N_i),\ i = 1, 2, \cdots, M, \tag{2.4}$$

which is often solved by the Newton method. Therefore the Jacobi matrix J of (2.4) is the main concern. The direct computation shows that

$$\{A(N_i, N_j) - (N_i, f'(\sum_{k=1}^{M} N_k U_k)N_j)\}_{M\times M}, \tag{2.5}$$

which has to be updated repeatedly as the iterations proceed. Obviously, the nonlinearity of $f(u)$ makes the integrations in the second term of (2.5) quite large in calculations and then results in the very expensive computation for the Newton method.

To overcome this difficulty, a simple and efficient solution called the interpolated coefficient finite element method (ICFEM), which was originally inspired by solving semilinear parabolic problems, was first proposed by M.Zlamal [11]. For implementation in this article, the details of the ICFEM are given as follows based on problem (2.1).

Substitute the interpolation $I_h f(u_h) = \sum\limits_{j=1}^{M} N_j(x) f(U_j)$ with $U_j = u_h(x_j)$ rather than $f(u_h)$ into (2.3) and still denote the ICFEM solution as $u_h = \sum\limits_{j=1}^{N} U_j N_j(x)$. Then, we obtain a new finite element equation

$$A(u_h, v) - (I_h f(u_h), v) = (g, v), v \in S^h. \tag{2.6}$$

As a result, we obtain a nonlinear algebraic system of equations

$$\sum_{j=1}^{M} (k_{ij} U_j - m_{ij} f(U_j)) = (g, N_i), i = 1, 2, \cdots, M, \tag{2.7}$$

where the elements of the stiffness matrix $k_{ij} = A(N_j, N_i,)$ and the elements of the mass matrix $m_{ij} = (N_j, N_i)$ can be computed once. The Jacobi matrix of (2.7) is

$$J_1 = \{k_{ij} - m_{ij} f'(U_j)\}_{M\times M}. \tag{2.8}$$

As k_{ij} and m_{ij} are given, the Jacobi matrix J_1 can be obtained simply by multiplying m_{ij} and $f'(U_j)$. Therefore, the computation is greatly reduced compared with that for solving (2.4).

Actually, the ICFEM has become one of the most efficient methods for solving a wide variety of semilinear elliptic or parabolic problems. For semilinear parabolic problems, some error analysis of ICFEM has been obtained (see [12], [13]). For the semilinear elliptic problem (2.1), [14] assumed

(**B1**) $u \in H^{n+1}(\Omega) \bigcap S_0$ is a solitary solution of (2.1), that is, $Q(u,v)=0$ and

$$|Q(w,v)| \geq c\|w-u\|\|v\|,\ w \in N_\epsilon(u),\ v \in S_0,\ c=c(u)>0,$$

where $N_\epsilon(u)=\{w|w \in H^{n+1} \bigcap S_0, \max_\Omega |u-w|<\epsilon\}$.

By introducing the auxiliary linear elliptic operator $B(w,v)=A(w,v)-(f'(u)w,v), v \in S_0$, [14] obtained the convergence of ICFEM for (2.1), that is.

Theorem 2.1 Suppose the assumption (B1) holds and the triangulation is quasiuniform. Then its n-degree ICFEM solution u_h has an optimal order convergence estimate $\|u_h-u\|=C(u)h^{n+1}$, where the constant $C(u)$ depends on the norm $\|u\|_{n+1,\Omega}$.

The proof can be seen in [14].

Remark 2.1 The condition $f'(u) \leq 0$, that is, the coerciveness of $B(w,v)$ is not required in Theorem 2.1. This fact means that the ICFEM can be used conveniently in the computation of the multiple solutions of (1.1), as will be seen later.

Actually, we are focusing on the superconvergence analysis of ICFEM and have obtained some promising results, e.g., the error estimate $O(h^4)$ at nodal points for the triangular and rectangular quadratic ICFEM with the uniform meshes. The corresponding work will be submitted in our subsequent articles.

3 The Improved Search-extension Method

As described in Section 1, the problem (1.1) associates with the functional $J(u)$ whose critical point u is a (weak) solution of (1.1), that is,

$$(\nabla u, \nabla v)-(f(u),v)=0,\ \ \forall v \in S_0. \tag{3.1}$$

Our present objective is to combine the FEM, the ICFEM, the eigenfunction extension method, and the SEM to overcome the difficulties caused by the general nonlinearity and domains to solve (1.1).

First consider the weak form of the eigenvalue problem w.r.t. $-\triangle$, that is,

$$(\nabla u, \nabla v)=\lambda(u,v),\ \ \forall v \in S_0, \tag{3.2}$$

whose eigenpairs $\{\lambda_j, \phi_j\}, j=1,2,3,\cdots$ can be obtained accurately in some symmetric domain Ω, e.g., a square or a circle. However, we have to apply the FEM to calculate them for the general domains or operators. Substituting $u=\sum\limits_{i=1}^{M} N_i V_i, v=N_j, j=1,2,\cdots,M$ in (3.2), we get

$$K_1 V=\lambda K_2 V, \tag{3.3}$$

where $K_1(i,j)=(\nabla N_i, \nabla N_j)$, $K_2(i,j)=(N_i,N_j)$, $i,j=1,2,\cdots,M$. Then we can obtain the eigenvalues λ and its corresponding eigenfunctions $\phi=\sum\limits_{i=1}^{M} V_i N_i(x)$ by the standard numerical method.

As the eigenfunctions of $-\triangle$ are only used in the search for the initial guesses in the ISEM below, the meshes for solving (3.2) by the FEM can be relatively crude to reduce the cost of the computation. However, the meshes for computing the multiple solutions of (1.1) by the ICFEM have to be much more refined to guarantee the accuracy. Then the number M of the interior nodes of the meshes varies accordingly.

Now we propose an algorithm to obtain the multiple solutions of semilinear elliptic PDES with general nonlinearity in general domains based on (1.1).

Improved Search-extension Algorithm

Step 1. Computing the eigenpairs of $-\triangle$.

Solve the eigenvalue problem (3.2) by the FEM if the eigenpairs are not obtained accurately.

Step 2. Search for the initial values in S_N.

Without loss of generality, assume that $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ and $\{\phi_j\}_1^N$ form a normalized orthogonal system, that is, $(\phi_i, \phi_j) = \delta_{ij}, A(\phi_i, \phi_j) = \lambda_j \delta_{ij}$. As $\int_\Omega F(u)\mathrm{d}x$ does not contain the derivative of $u \in H^1$, it possesses some compact property. Set $S_N = \mathrm{span}\{\phi_1, \phi_2, \cdots, \phi_N\}$. Then the solutions of (1.1) can be approximated by

$$u(x) = \sum_{j=1}^{N} a_j \phi_j(x) \in S_N, \tag{3.3}$$

with N being large enough. To determine $\mathbf{a} = \mathbf{a}(N) = [a_1, a_2, \cdots, a_N]^{\mathrm{T}}$, substituting (3.3) into $J(u)$, we obtain

$$J(u) = \frac{1}{2} \sum_{j=1}^{N} \lambda_j a_j^2 - \int_\Omega F(u)\mathrm{d}x,$$

whose critical points satisfy

$$\frac{\partial J(u)}{\partial a_i} = F_i(\mathbf{a}) = \lambda_i a_i - g_i(\mathbf{a}) = 0, \quad g_i(\mathbf{a}) = (f(\sum_{j=1}^{N} a_j \phi_j), \phi_i), \ i = 1, 2, 3, \cdots, N. \tag{3.4}$$

This is an algebraic system of equations.

Suppose λ_l is a k-tuple eigenvalue and the corresponding eigenfunctions ϕ_l, $\phi_{l+1}, \cdots$, ϕ_{l+k-1} span the subspace S_k^*. We may consider that all nonzero solutions $u_l(x)$ of (1.1) are arranged in the order of eigenvalues λ_l. By taking N appropriately large such that $S_N \supset S_k^*$, the solutions of (3.4) $\mathbf{a^0} = \mathbf{a^0}(N) = [a_1^0, a_2^0, \cdots, a_N^0]^{\mathrm{T}}$ can be searched out and the rough initial guesses of the solutions u_l are obtained. Indeed, in many simpler cases, we only need to search for the initial guesses of the solutions in S_k^*. In this manner, we obtain a system of algebraic equations with k unknowns $\mathbf{a}(k) = [a_l, a_{l+1}, \cdots, a_{l+k-1}]^T$, that is,

$$\lambda_i a_i = g_i(\mathbf{a}(k)), \quad i = l, l+1, \cdots, l+k-1. \tag{3.5}$$

When k is not large, $\mathbf{a}(k)$ can be searched out by a simple algorithm. Therefore, we get a rough initial approximation $u_l^0 \approx \omega \sum\limits_{j=l}^{l+k-1} a_j^0 \phi_j$ for each root $\mathbf{a}^0(k)$, where $\omega \in (0.5, 1]$ can be chosen in computation. Actually we always take $\omega = 1$ at first, it will do if the numerical results are convergent. Otherwise we will reduce ω gradually until the results are convergent. In some complicated cases, we should increase the number of bases in order to search for more and better initial values. The purpose of this step is to separate all the solutions and determine their rough positions.

Step 3. Discretize (1.1) by the ICFEM.

By the ICFEM, the discrete form of (3.1) is

$$K_1U - K_2F_1(U) = 0, \tag{3.6}$$

where $F_1(U) = [f(U_1), f(U_2), \cdots, f(U_M)]^{\mathrm{T}}$.

Step 4. Solve (3.6) by the numerical extension method.

Set $F(U) = K_1U - K_2F_1(U)$, whose Jacobi matrix is $J(U) = K_1 - K_2\mathrm{diag}(f'(U_1), f'(U_2), \cdots, f'(U_M))$, and $G(U) = DF(U^0)(U - U^0)^T$ with $U^0(i) = u_l^0(i), i = 1, 2, \cdots, M$ where u_l^0 is obtained in step 2. Then define an extension vector function

$$H(U,t) = tF(U) + (1-t)G(U), \ \ 0 \le t \le 1. \tag{3.7}$$

Instead of solving (3.6) directly, we will solve the homotopy equation

$$H(U,t) = 0, \ \ 0 \le t \le 1 \tag{3.8}$$

by the Newton method. When $t = 0$, (3.8) becomes $G(U) = 0$ whose solution vector is U_0. When $t = 1$, (3.8) becomes $F(U) = 0$, which is the target equation.

To solve (3.8), we take a subdivision $t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1$, where $t_i = i/m, i = 0, 1, 2, \cdots, m$, and m is large enough, and solve successively each nonlinear subproblem

$$\mathbf{P_i}: \quad \mathbf{H}(\mathbf{U}(t_i), t_i) = 0, \ \ i = 1, 2, \cdots, m. \tag{3.9}$$

To solve the subproblem $\mathbf{P_i}$, the Newton method is used by taking the approximate solution for $\mathbf{P_{i-1}}$ as its initial guess except that the initial guess is U^0 for problem $\mathbf{P}_1$. In solving each subproblem (3.9) with $i = 1, 2, \cdots, m-1$, we use the Newton method several times until $\|U^{(p)}(t_i) - U^{(p-1)}(t_i)\| < \epsilon_0$ for some $p \ge 2$. Finally we solve $\mathbf{P}_m$, that is, the target equation (3.6) by the Newton method with a pretty good initial guess until

$$\|U^{(p)}(1) - U^{(p-1)}(1)\| < \epsilon_1, \tag{3.10}$$

where ϵ_1 is much less than ϵ_0.

Remark 3.1 The most important advantages of the improved search-extension algorithm (ISEM) are:

(i) In Step 1, when Ω is some symmetric domain, e.g., a 1-D interval, a square or a circle, the eigenvalue problem can be solved accurately. However, for the general elliptic operator and domain, the FEM is an efficient tool to solve the corresponding eigenvalue problem. This step makes it possible for our approach to be used for solving a broad type of semilinear elliptic problem in various domains.

(ii) In Step 2, the linear combination of some eigenfunctions corresponding to the k-tuple eigenvalue λ_l supplies a reasonable initial guess for the solutions of (1.1). In such a manner, we can deduce that the solutions of (1.1) are distributed in terms of the eigenvalues of $-\triangle$ and gain some knowledge about the structure and distribution of the solutions.

(iii) In Step 3, the ICFEM reduces the computation works greatly.

(iv) In Step 4, the numerical extension method guarantees the global convergence of our approach.

4 The Computational Examples

When $f(u)$ is of non-odd nonlinearity, only three solutions of (1.1) in general domains were verified to exist theoretically and at most four solutions were computed numerically as described in Section 1. Based on the problem

$$\triangle u + f(u) = 0 \ \text{ in } \ \Omega, \ u = 0 \ \text{ on } \ \partial\Omega, \tag{4.1}$$

where

$$f(x,u) = \begin{cases} u^3, & \text{if} \quad u \geq 0, \\ u^5, & \text{if} \quad u \leq 0, \end{cases}$$

and $\Omega \subset R^2$ are, respectively, a square, a triangle, and a concave L-type domain, our numerical results will show that the ISEM is robust for the problems with non-odd nonlinearity.

Case 1 Let $\Omega = \Omega_s = (0,\pi) \times (0,\pi) \subset R^2$.

In this case, the eigenpairs of $-\triangle$ are $\{\lambda_{p,q}, \phi_{p,q}\}$ with $\lambda_{p,q} = p^2 + q^2$, $\phi_{p,q} = \sin px_1 \sin qx_2$, $p,\ q = 1,2,3,\cdots$. We have computed the solutions with the initial guesses that are the linear combinations of the eigenfunctions corresponding to the single eigenvalues $\lambda_{1,1}$ and $\lambda_{3,3}$ and the double eigenvalues $\lambda_{1,2}$ and $\lambda_{1,3}$, including the positive and negative mountain pass solutions. For simplicity, we present only two of them in Fig. 1–2 and Table 1.

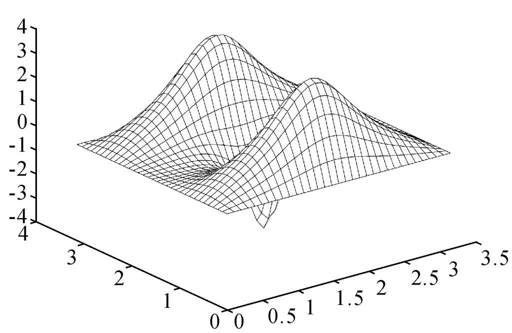

Fig.1 A solution in Ω_s with the initial guess $a\phi_{1,3} + b\phi_{3,1}$

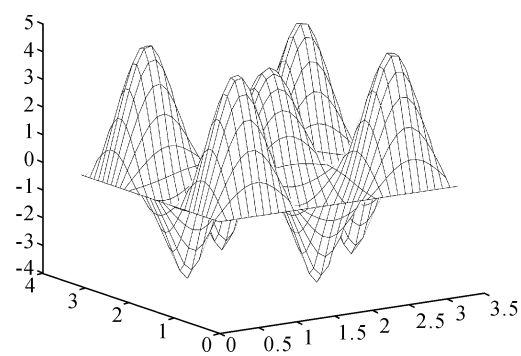

Fig.2 A solution in Ω_s with the initial guess $a\phi_{3,3}$

Table 1 The solutions of (4.1) in Ω_s

The initial guess	Eigenvalueλ	$u_{\max}$	$u_{\min}$
$a\phi_{1,3} + b\phi_{3,1}$	10	3.8242	-3.2999
$a\phi_{3,3}$	18	4.7261	-3.0847

We note that Fig. 1–2 are nonsymmetric because of the non-odd nonlinearity of $f(u)$.

Case 2 Let $\Omega = \Omega_t = \{(x_1,x_2) \in R^2 | x_1 > 0, x_2 > 0, x_1 + x_2 \leq 1\}$.

In this case, the domain is nonsymmetric, so Step 1 of the ISEM is used to determine the eigenpairs of $-\triangle$ numerically and then we can obtain the solutions of (4.1) in Ω_t by the ISEM. Indeed, we have computed eight solutions of (4.1) in Ω_t with the initial guesses $a\phi_i$, $i = 1,2,3,4$, including the positive and negative mountain pass solutions. Subsequently, four of these are shown in Fig. 3–6 and Table 2.

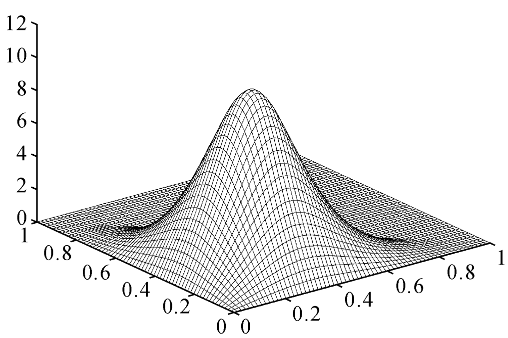

Fig.3 The positive mountain pass solution in Ω_t with the initial guess $a\phi_1$

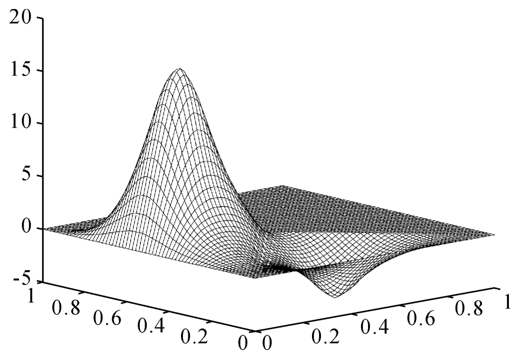

Fig.4 A solution in Ω_t with the initial guess $a\phi_2$

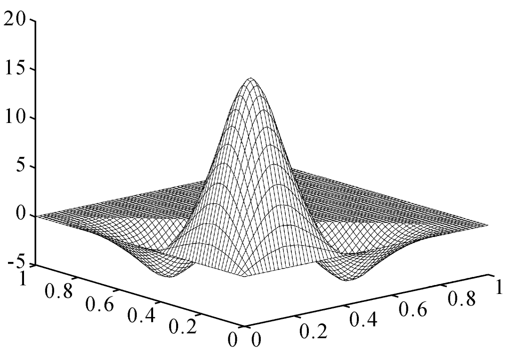

Fig.5 A solution in Ω_t with the initial guess $a\phi_3$

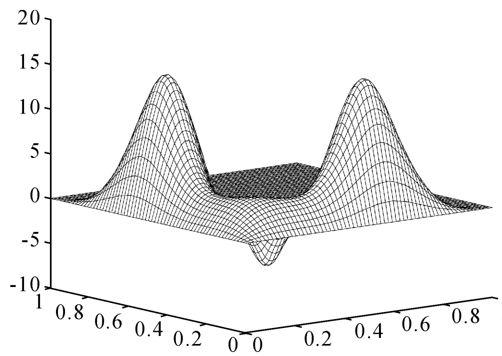

Fig.6 A solution in Ω_t with the initial guess $a\phi_4$

Table 2 The solutions of (4.1) in Ω_t

The initial guess	Eigenvalue λ	$u_{\max}$	$u_{\min}$
$a\phi_1$	49.3506	10.5494	0
$a\phi_2$	98.7212	16.2363	-4.5372
$a\phi_3$	128.3519	18.3905	-4.8468
$a\phi_4$	167.8930	15.2311	-6.1799

Case 3 Let $\Omega = \Omega_L = \{(x_1, x_2) \in R^2 | [-1,1] \times [0,1] \cup [-1,0] \times [-1,0]\}$, which is an L-type domain.

In this case, the domain is concave and the eigenpairs of $-\triangle$ are unknown. Similar to Case 2, we can determine the eigenpairs of $-\triangle$ numerically and then obtain the solutions of (4.1) in Ω_L by the ISEM.

Indeed, we have obtained six solutions with the initial guesses $a\phi_i$, $i = 1, 2, 3$. Furthermore, we have also obtained the solutions with the initial guesses $a\phi_8 + b\phi_9$ corresponding to the least double eigenvalues $\lambda_8 = \lambda_9$. Here a and b are also determined by Step 2 in the ISEM. For simplicity, we shall show below four of these in Fig. 7–10 and Table 3.

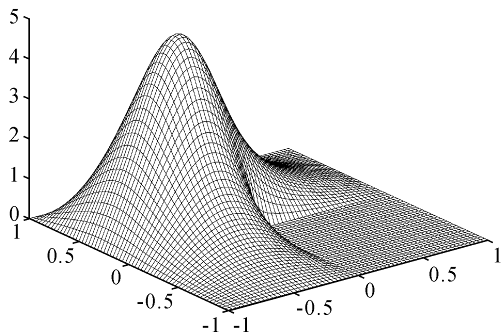

Fig.7 A positive mountain pass solution in Ω_L with the initial guess $a\phi_1$

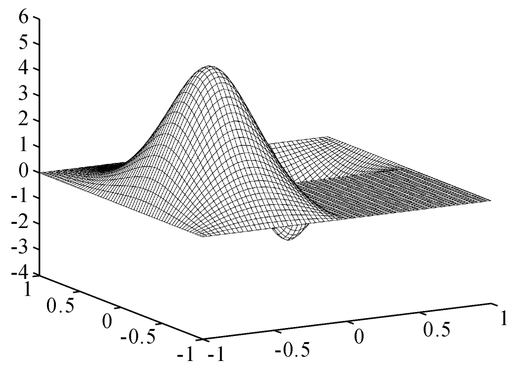

Fig.8 A solution in Ω_L with the initial guess $a\phi_2$

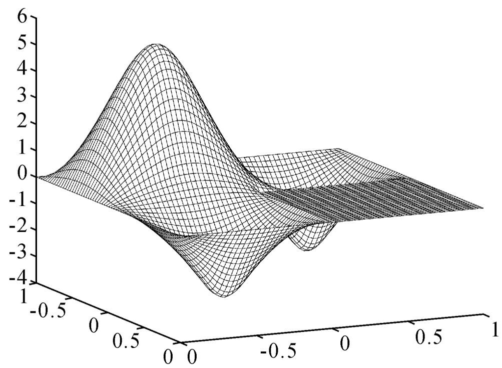

Fig.9 A solution in Ω_L with the initial guess $a\phi_3$

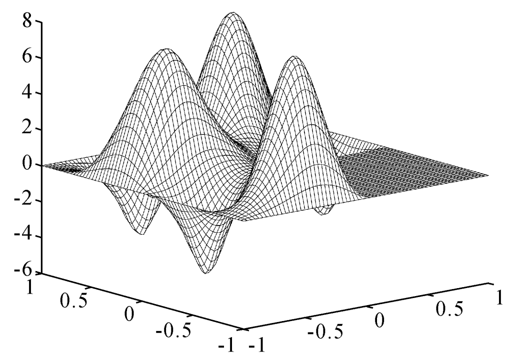

Fig.10 A solution in Ω_L with the initial guess $a\phi_8 + b\phi_9$

Table 3 The solutions of (4.1) in Ω_L

The initial guess	Eigenvalue λ	u_{max}	u_{min}
$a_1\phi_1$	9.6731	4.8739	0
$a_2\phi_2$	15.2083	5.3523	-3.1025
$a_3\phi_3$	19.7399	5.5854	-3.3610
$a_8\phi_8 + b_9\phi_9$	49.3686	7.8396	-4.3075

5 Some Further Discussions

In Section 3, we have proposed a new algorithm (ISEM) for computing the solutions of semilinear elliptic equation (1.1). As the theoretical analysis and numerical computation of the multiple solutions for (1.1), in which $f(u)$ is of non-odd nonlinearity, encounter the inherent difficulties, as shown in Section 1, we have implemented the ISEM to a typical model problem (4.1) in various domains. We note that the ISEM can obtain more than eight solutions with two of them, the positive and negative mountain pass solutions, in the general domains Ω. Indeed, for a square and a concave L-type domain, we have obtained more than 20 solutions for (4.1). Nevertheless, we encountered some difficulties in the numerical computation for (4.1) in a triangle when the eigenvalues are large. By numerical experiments implementing Matlab software, we note that the corresponding eigenvalues increase rapidly and the eigenpairs with large eigenvalues cannot be computed accurately enough. We guess that this may result in the inaccuracy of the initial guess determined in Step 2. Therefore, we can at least prove that (1.1) has much more than three solutions when $f(u)$ is non-odd nonlinearity. Actually, we have computed the solutions of other problems of type (1.1) with non-odd nonlinearity, we have a conjecture that is as follows.

Conjecture If $f(u)$ satisfies the hypotheses (A1)–(A4) and $\lim\limits_{|u|\to\infty} \frac{f(x,u)}{u} = +\infty$, then (1.1) has an infinite number of solutions.

Moreover, the fact that the solutions obtained by the ISEM are listed in terms of the eigenvalues describes the structure and distribution of the solutions. On the other hand, it is known that the Morse index is an important notation that provides understanding of the local structure of a saddle point and can be used to measure the instability of a saddle point. The first two authors have developed a numerical method to compute the Morse indices of the solutions of (1.1) in one-dimensional domains. In a subsequent article, we will compute the Morse indices of the solutions of (1.1) in two-dimensional domains implementing the analogous strategy of that in [15].

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