Modified Minkowski Metric Spacetime and Momenergy Diagrams Simplify Relativity – Part 1

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Abstract

A modification is proposed in the Minkowski spacetime metric, whose associated geometry defines 4-dimensional spacetime in Einstein's special theory of relativity. The invariant spacetime interval quantity $s=c\tau$ replaces the locally measured time quantity ct in the Minkowski spacetime 4-vector, which also includes the three spatial dimensions x, y and z. Similarly, the associated invariant-mass-related quantity mc replaces the locally measured total-energy-related quantity E/c in the Minkowski momenergy 4-vector. The former Minkowski metric coordinates ct and E/c become simple hypotenuse lengths in this new approach. These modifications significantly simplify the geometry of the Minkowski spacetime and momenergy diagrams used with special relativity without changing the special relativity equations themselves. Use of the Modified Minkowski metric could therefore greatly simplify the teaching and understanding of special relativity and invariance, which now heavily relies on Minkowski spacetime and momenergy diagrams.

Key words: Minkowski diagram, spacetime interval, momenergy, relativity, worldline, spacetime event, invariant mass, time, space, mass, energy

Introduction

According to Einstein's[1] special theory of relativity, lengths and time durations measured for the same occurrence such as a passing rocket ship or a passing sub-atomic particle may be found to be different for different observers moving with a constant velocity relative to each other. The speed of light measured by all observers, independent of their state of motion, is the constant value of approximately $c = 3.00\times10^{8}\,\text{m/s}$. Minkowski[2] in his lecture "Space and Time" proposed a 4-dimensional geometrical spacetime metric that fits Einstein's special relativity equations for the invariance relationships found from measurements of distances and times in different inertially-moving coordinate systems. In the Minkowski metric, spacetime is geometrically flat and objects move in straight lines when not acted on by outside forces. The Minkowski spacetime and momenergy metric approach has been widely used in teaching special relativity, and has also stimulated discussion about the nature of space and time.

Although Minkowski's spacetime metric was developed for use with special relativity, the metric is also used in Einstein's[3] general theory of relativity, which describes gravity as curved spacetime. The curved spacetime metric obtained from Einstein's general relativity field equations can be described on small-enough scales as following the Minkowski metric.

The Minkowski Spacetime Metric and the Event

An event E = (ct, x, y, z) is a mathematical point representing a set of coordinates in space and time. The measured time t is multiplied by c to give units of length to all four coordinates. The corresponding Minkowski spacetime metric or 4-vector is $X=(ct,x,y,z)$. The Minkowski metric X contains the same values as the event E. The Minkowski metric is used for calculating time changes $c\Delta t$ (measured in distance units), position changes Δx and invariant spacetime intervals Δs. These quantities are pictured on Minkowski spacetime diagrams. Since a 4-dimensional diagram cannot be drawn accurately in two dimensions, Minkowski spacetime diagrams are usually drawn (without loss of generality) showing one or two dimensions for positions such as x and y, and one dimension for time t, depicted as ct to give this dimension the same length units as x, y and z for the positions of an event in space and time. Figure 1 shows a Minkowski 4-vector $X=(ct_o,x_o,y_o,z_o)=(5\text{m},6\text{m},0,0)$, where the z-axis is not shown.

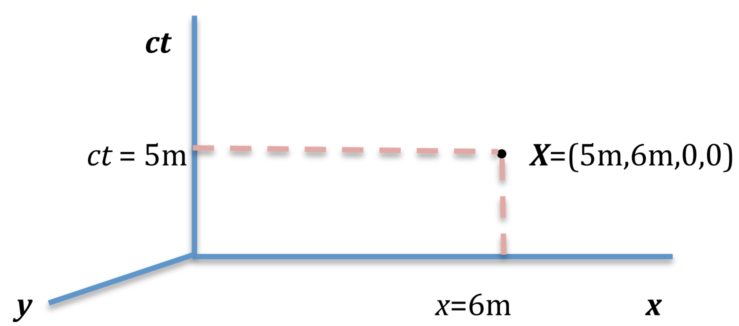

Figure 1. A Minkowski 4-vector in spacetime where ***X***=(5m,6m,0,0) as measured from one inertial reference frame. The z-axis is not shown.

The Minkowski Invariant Spacetime Interval Δ*s* and the Spacetime Diagram

Minkowski[2] showed that Einstein's space and time transformations can be represented geometrically in a spacetime world. In this mathematical world, invariant spacetime intervals Δs are found using the Minkowski metric X=(ct,x,y,z) when the space and time coordinates of two events E_1 and E_2 are measured by an observer in an inertially moving coordinate system. Observers measuring two events in different inertially moving coordinate systems, will calculate the same spacetime interval Δs between the two events, even though the event coordinates are themselves are quantitatively different as measured from different inertial coordinate systems. This is why Δs is called an invariant quantity.

The Minkowski metric labels a spacetime event as $E=(ct,x,y,z)$ where t, x, y, and z are the measured values for the event in an inertial coordinate system moving with a particular constant velocity. For two such events $E_1=(ct_1,x_1,y_1,z_1)$ and $E_2=(ct_2,x_2,y_2,z_2)$, the corresponding Minkowski 4-vectors are $X_1=(ct_1,x_1,y_1,z_1)$ and $X_2=(ct_2,x_2,y_2,z_2)$. The spacetime interval Δs defined by $(\Delta s)^2=(c\Delta\tau)^2=(X_2-X_1)^2=(c\Delta t)^2-(\Delta x)^2-(\Delta y)^s-(\Delta z)^2$ is found according to special relativity to be invariant for the same two events as measured from any inertially moving coordinate system. The quantity $\Delta\tau$ is called the "proper time" corresponding the spacetime interval Δs. To illustrate these mathematical relationships, Minkowski produced spacetime diagrams corresponding to his metric. These diagrams show graphically and geometrically the relativistic *ct*, *x*, *y* and *z* relationships and the invariant spacetime interval Δ*s*. These diagrams are called Minkowski diagrams.

Unfortunately the invariant spacetime interval Δ*s* is not the actual hypotenuse in a Minkowski spacetime diagram. This is because the *ct* and *x (*and *y* and *z)* axes are orthogonal in Minkowski spacetime diagrams. In Figure 2 for example, the event E_1 equals (0,0,0,0) and the event E_2 equals (5m,4m,0,0) . So the Minkowski 4-vectors are $X_1 = (0,0,0,0)$ and $X_2 = (5m,4m,0,0)$. $X_2 - X_1 = (c\Delta t, \Delta x, \Delta y, \Delta z) = (5m,4m,0,0)$. So $\Delta s = \sqrt{(c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2} = \sqrt{5^2 - 4^2 - 0^2 - 0^2} = 3m$. Here Δ*s* is not $\sqrt{5^2 + 4^2} = \sqrt{41} = 6.4\text{m}$ which is the length of the hypotenuse in Figure 2. Following Taylor and Wheeler[4], the invariant interval *Δs* = 3m is shown along the diagonal of the right triangle formed by the differences *cΔt* and Δ*x* (in a 2-dimensional Minkowski spacetime diagram where Δy=Δz=0). See Figure 2.

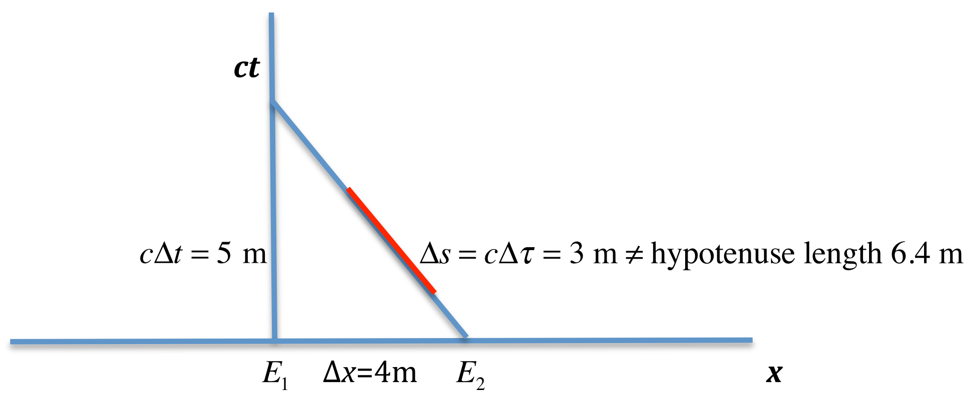

Figure 2. The invariant interval Δs (in red) between two events E_1 and E_2 is shown in a Minkowski spacetime diagram, following Taylor and Wheeler[4]. The length of $\Delta s = \sqrt{(c\Delta t)^2 - (\Delta x)^2}$ is not equal to the triangle's hypotenuse.

The Modified Minkowski Metric and Spacetime Diagram

In this article I propose a Modified Minkowski metric with associated Modified Minkowski spacetime diagrams having geometrical simplicity lacking in standard Minkowski spacetime diagrams. An event *E* = (*ct,x,y,z*) is defined the same as before. The Modified Minkowski spacetime metric replaces the first coordinate *ct* in the Minkowski metric $X = (ct,x,y,z)$ by the spacetime interval coordinate *s=cτ*, forming the Modified Minkowski metric $X = (c\tau,x,y,z)$. Einstein's invariant interval equation $(\Delta s)^2 = (c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$ is then generated from the Modified

Minkowski metric *X* using the regular Pythagorean hypotenuse length formula for four dimensions: $(X_2 - X_1)^2 = (c\Delta\tau)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (c\Delta t)^2$, which is equivalent to Einstein's invariant spacetime interval equation above. The Modified Minkowski spacetime metric gives a Modified Minkowski spacetime diagram whose vertical axis $s = c\tau$ is orthogonal to the *x*, *y* and *z*-axes. (Modified Minkowski diagrams in this article show only the $c\tau$ axis and the *x*-axis or the *x* and *y*-axes.) In Modified Minkowski spacetime diagrams the intervals $c\Delta\tau$, $c\Delta t$, Δx, Δy and Δz are related geometrically by a right triangle whose hypotenuse is $c\Delta t$, whose vertical side is $c\Delta\tau$ and whose horizontal side is Δx (standing without loss of generality for the three spatial dimensions *x*, *y* and *z*, which cannot all be accurately shown together along with the $c\tau$-axis on a 2-dimensional diagram.)

Figure 3 shows a Modified Minkowski spacetime diagram showing the same invariant interval $\Delta s = c\Delta\tau$ = 3m as in Figure 2, and where $c\Delta t$ = 5m and Δx = 4m as in Figure 2. The events are indicated at the *x* values where they occur, such as E_1 and E_2 in Figure 3.

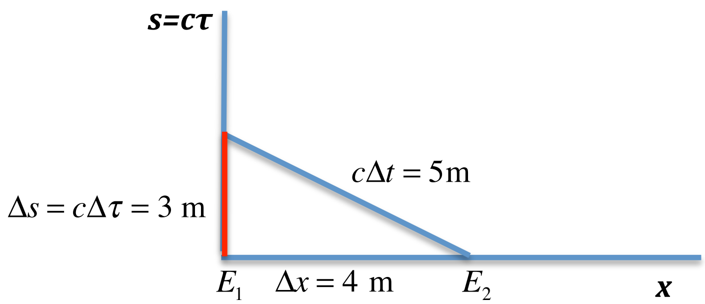

Figure 3. A Modified Minkowski spacetime diagram shows same invariant spacetime interval Δs (in red) as in Figure 2. Now the value of $\Delta s = (c\Delta t)^2 - (\Delta x)^2$ is equal to the length of the right triangle's vertical side. The events $E_1 = (0,0,0,0)$ and $E_2 = (5m, 4m, 0, 0)$ on which the calculations are based are indicated in the diagram.

The Minkowski Momenergy 4-Vector and Diagram for a Single Object with Mass

The invariant mass m of a relativistically moving object (or set of objects considered as a single system) is given in the experimentally well-established relativistic energy-momentum equation $E^2 = p^2c^2 + m^2c^4$, where the object or system's total energy E and total linear momentum p are measured in any inertial frame, and the mass *m* is an invariant quantity, i.e. it is independent of the inertial system from which *E* and *p* were measured for the object or set of objects. An inertial frame is a reference frame that is moving with a constant velocity relative to the reference frame where the object or system of objects has zero total momentum. The standard Minskowski momentum 4-vector $P = (E/c, p_x, p_y, p_z)$ yields the above energy-momentum equation $E^2 = p^2c^2 + m^2c^4$. But, similar to the Minkowski spacetime calculation for Δs, the squares of the last three momentum components are subtracted from the square of the first momentum component *E/c*. This gives

$P^2 = (E/c)^2 - p_x{}^2 - p_y{}^2 - p_z{}^2 = (E/c)^2 - p^2 = (mc)^2$ which is the relativistic energy-momentum equation.

In 2-dimensional Minkowski momenergy diagrams, the object's total energy E is plotted as the momentum *E/c* on the vertical axis. The momentum component p_x (standing without loss of generality for the momentum coordinates p_x, p_y and p_z) is plotted on the horizontal axis. The invariant momentum quantity *mc* is indicated, following Taylor and Wheeler[4], by a short length (in red) along the hypotenuse of the resulting right triangle. In Figure 4 below, the Minkowski momentum 4-vector *P* = (5 MeV/*c*, 4 MeV/*c*, 0, 0) is diagramed. The length of the hypotenuse is $\sqrt{5^2 + 4^2} = \sqrt{41} = 6.2 \text{ MeV}/c$, while the invariant momentum *mc* is 3 MeV/*c.*

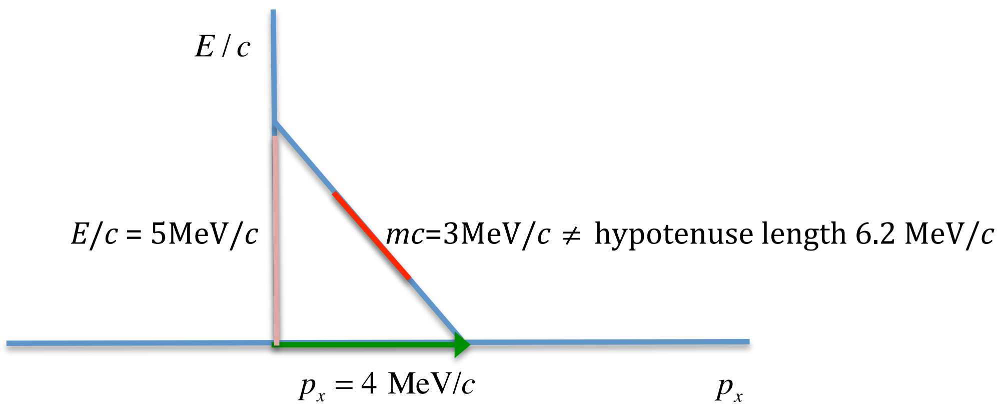

Figure 4. A Minkowski 4-momentum diagram shows the invariant momentum *mc* (in red) for Minkowski 4-vector *P* = (5 MeV/*c*, 4 MeV/*c*, 0,0). The value of this invariant momentum $mc = \sqrt{(E/c)^2 = p_x{}^2}$ is less than the corresponding triangle's hypotenuse, following Taylor and Wheeler[4] . The invariant mass *m* is found from this invariant momentum *mc*.

The Modified Minkowski Momentum 4-Vector and Diagram for a Single Object with Mass

For the Modified Minkowski 4-momentum metric the invariant momentum *mc* replaces the momentum *E/c* of the Minkowski metric, giving $P = (mc, p_x, p_y, p_z)$ rather than $P = (E/c, p_x, p_y, p_z)$. The amplitude of the Modified Minkowski 4-vector *P* is then calculated as $P^2 = (mc)^2 + p_x{}^2 + p_y{}^2 + p_z{}^2 = (mc)^2 + p^2 = (E/c)^2$ using the standard 4-d Pythagorean hypotenuse length formula. So in this way the Modified Minkowski momentum 4-vector generates the correct relativistic energy-momentum equation $E^2 = p^2c^2 + m^2c^4$, as does the Minkowski momentum 4-vector.

Figure 5 shows a Modified Minkowski momenergy diagram with the same invariant momentum *mc* = 3 MeV/c as in Figure 4, but now as the vertical side of the triangle.

The length of the hypotenuse corresponds to the momentum quantity E/c = 5 MeV/c of the particle, and $p_x = 4$ MeV/c , while $mc = \sqrt{5^2 - 4^2} = 3$ MeV/c as before.

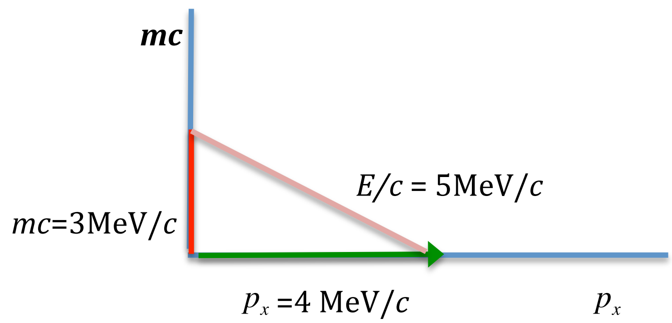

Figure 5. The same invariant momentum mc shown (in red) on a modified Minkowski 4-momentum diagram. The value of the invariant quantity $mc = \sqrt{(E/c)^2 - p_x^2}$ is equal to the right triangle's vertical side length.

Figure 6(a) shows a Minkowski momenergy diagram for a single particle having invariant momentum mc = 3.0 MeV/c (and invariant mass $m = 3.0$ MeV/c^2), as measured from two different inertial frames moving in the x-direction. In one inertial frame, E/c equals 5.0 MeV/c and the momentum component p_x equals 4.0 MeV/c. In a second inertial frame moving with a constant velocity relative to the first, the particle's E/c equals 6.0 MeV/c and its p_x equals 5.2 MeV/c. Figure 6(b) shows a Modified Minkowki momenergy diagram with the same information. (Assume that $p_y = p_z = 0$ in both inertial systems.) The reader can decide which diagram showing the particle's invariant momentum mc=3.0 MeV/c is simpler.

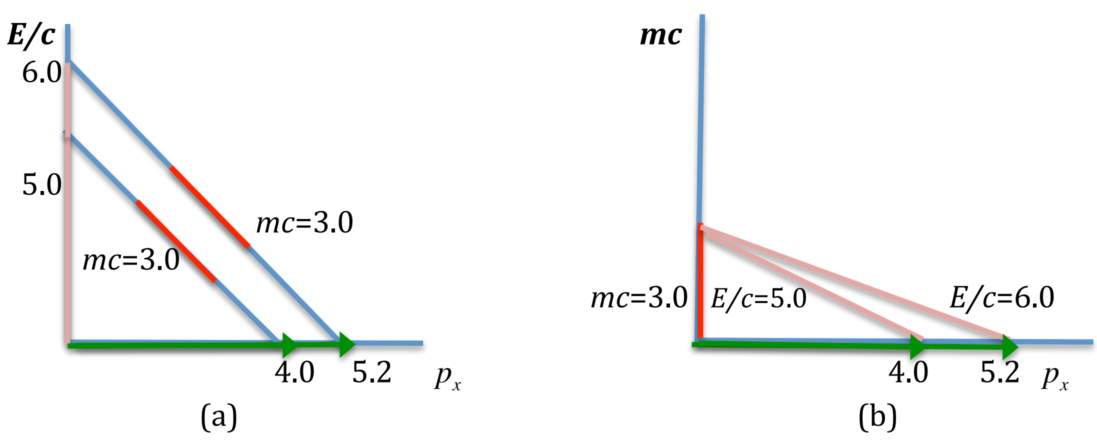

Figure 6(a): Minkowski diagram shows a particle having invariant momentum mc (and invariant mass m) but with two different energy quantities E/c and two different momentum components p_x as measured from two different inertial frames. Figure 6(b): Modified Minkowski diagram shows the same particle information measured from the same two different inertial frames as in Figure 6(a). Which diagram is simpler?

The Modified Minkowski spacetime diagram for a photon

Figure 7 below shows the representation of a photon in the Minkowski and Modified Minkowski spacetime diagrams. The Minkowski diagram on the left, Figure 7(a), shows the Minkowski 4-d spacetime vector ***X***=($c\Delta t$, Δx, 0,0) for a photon, here seen moving in the +*x* direction. The photon has $c\Delta t$ equal to Δx so it forms a 45-degree angle in the ***ct-x*** plane. The Modified Minkowski diagram on the right, Figure 7(b), shows the same photon with Modified Minkowski 4-vector *X*=(0,Δx=$c\Delta t$,0,0). In this Modified Minkowski diagram, the photon's spacetime interval cΔτ is zero on the vertical **cτ** axis but the photon's displacement Δx equals $c\Delta t$ on the horizontal ***x*** axis.

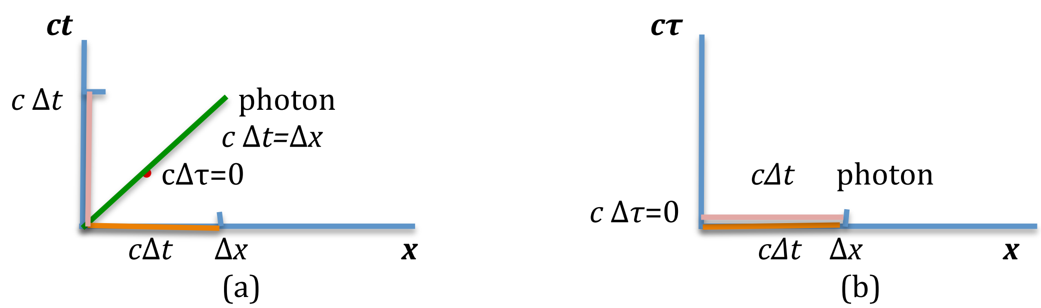

Figure 7. Comparison of (a) the Minkowski spacetime diagram and (b) the Modified Minkowski spacetime diagram for a single photon. The spacetime interval $c\,\Delta\tau$ for a photon is always zero, as indicated by the red spot on the green hypotenuse line in Figure 7(a) and the 0 value of $c\Delta\tau$ on the ***cτ*** axis in Figure 7(b).

The Modified Minkowski momenergy diagram for a photon

Figure 8 below shows the representations of a photon in Minkowski and Modified Minkowski momenergy diagrams. The left diagram shows the Minkowski momentum 4- vector *P*=(E/c, p_x ,0,0) for a photon, here seen moving in the +x direction. The photon has E/c equal to its momentum component p_x (assuming $p_y = p_z = 0$) and so forms a 45-degree angle in the ***E/c*** - p_x plane. The right diagram shows the Modified Minkowski momentum 4-vector *P*=(0, $p_x = E/c$,0,0). Its invariant momentum *mc* is zero as shown on the vertical ***mc*** axis. This is why the mass of a photon is said to be zero. But the photon's momentum component is $p_x = E/c$ on the horizontal p_x axis.

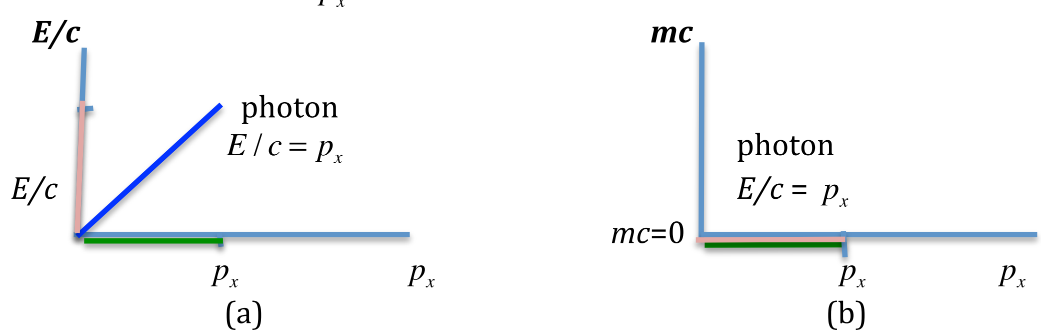

Figure 8. Comparison of (a) Minkowski momenergy diagram and (b) Modified Minkowski momenergy diagram for a single photon. The invariant momenergy *mc* for a photon is always zero. This is why the mass *m* of a photon is said to be zero.

Minkowski spacetime light cones and worldlines

Figure 9 shows the familiar Minkowski spacetime light-cone and worldline diagram indicating the timelike (v < c) and spacelike (v > c) regions on opposite sides of two light cones that are themselves the light-like region, allowing speed-of-light (v = c) travel along the sides of the cones.

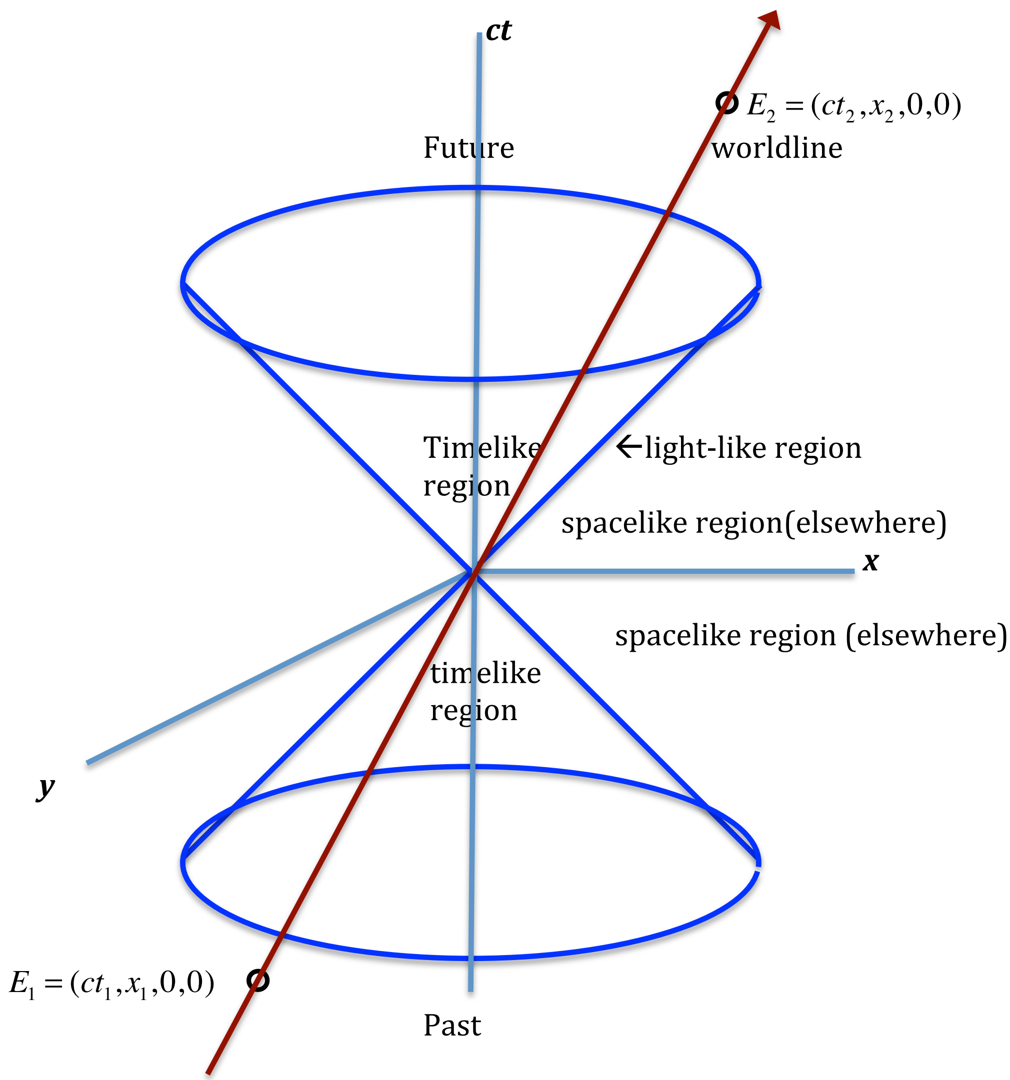

Figure 9. A Minkowski spacetime light cone diagram shows the different causal regions corresponding to an event E= (0,0,0,0) at the origin. The timelike regions (inside the light cones) are regions where there can be a causal relationship between the event at the origin and an event within the timelike regions (where v < c.) The spacelike regions (outside the light cones) are regions where there cannot be a causal relationship between the event at the origin and any event in the spacelike region (because $v > c$.) A worldline is shown for an object with mass moving at a constant speed $v < c$. Worldlines of objects with mass can move only through timelike regions. Events in light-like regions (the two cones themselves) can also be causally connected by lightspeed photon transfer.

Figure 10 below shows the Modified Minkowski spacetime diagram for a situation similar to that shown in the Minkowski light cone spacetime diagram above. Notice the absence of light cones in Figure 10. There are no light cones in Modified Minkowski spacetime diagrams because for light, $\Delta s = c\Delta\tau$ is always zero on the vertical axis. The worldline is shown for an object with mass moving with a constant velocity $v < c$ in the positive x direction. The spacetime interval $c\Delta\tau$ between the two timelike events $E_1 = (ct_1, x_1, 0, 0$ and $E_2 = (ct_2, x_2, 0, 0)$ is calculated as $c\Delta\tau = \sqrt{(c\Delta t)^2 - (\Delta x)^2}$ and is always real and positive when the object's velocity v is less than c. In the Modified Minkowski diagram, relations between two events can be causal as long as $\Delta s = c\Delta\tau$ is real and positive between these two events.

The spacetime interval $\Delta s = c\Delta\tau$ would be imaginary (or spacelike) if the velocity $v = \Delta x/\Delta t$ connecting two events separated by Δx and Δt is greater than c. Events with spacelike separation would yield an imaginary value of $\Delta s = c\Delta\tau$ and so could not have an associated worldline (see example in upper left quadrant in Figure 10.)

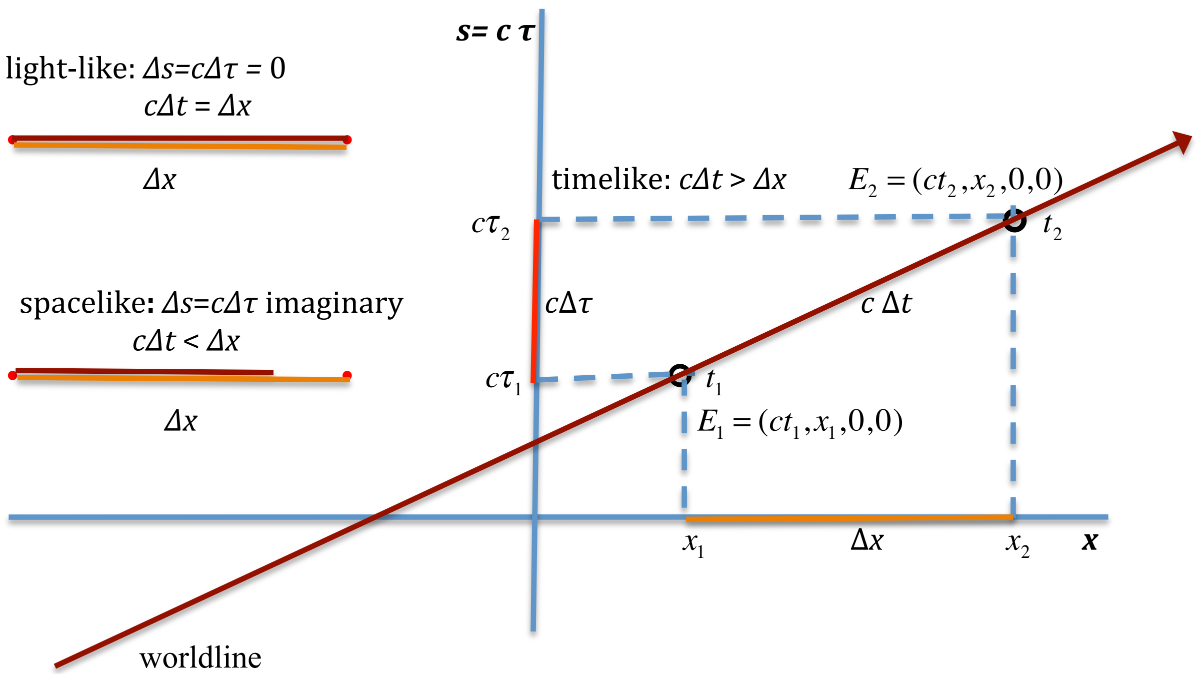

Figure 10. A Modified Minkowski spacetime diagram is shown with ***cτ*** and ***x*** orthogonal dimensions. An object is shown moving with a constant velocity v=*Δx/Δt* on its worldline, where *v*<*c*. The spacetime interval $\Delta s = c\Delta\tau$ between the two events $E_1 = (ct_1, x_1, 0, 0)$ and $E_2 = (ct_2, x_2, 0, 0)$ is calculated as $c\Delta\tau = \sqrt{(c\Delta t)^2 - (\Delta x)^2}$. On a Modified Minkowski spacetime diagram the worldline of a photon is always horizontal (slope=0), but the worldline of a material object with mass *m* moving at *v*<*c* will always have a slope greater than 0. This means that *cτ* is always increasing along the material object's worldline. If $\Delta x/c\Delta t > 1$ then the relation between two events is spacelike and there could be no causal relationship between these two events. A spacelike situation is shown by two horizontal line segments in the upper left quadrant above. The time separation $c\Delta t$ (length of the top line segment) between the two events is less than the distance separation Δx (length of the bottom line segment) between the two events. There can be no causal relation between these two events that are indicated by the two red dots. A lightlike region corresponds a spacetime interval Δs where $\Delta x = c\Delta t$, and $\Delta s = c\Delta\tau = 0$, also shown in the upper left quadrant of Figure 10.

Modified Minkowski Momenergy Diagrams for System of Two Particles Moving in Opposite Directions

Figure 11 show Modified Minkowski momenergy diagrams for two particles each with mass $m = 3\ \text{MeV}/c^2$ moving in opposite directions with momentum $p_{x1} = -4\text{MeV}/c$ and $P_{x2} = +4\text{MeV}/c$ on the p_x axis (with $p_y = p_z = 0$ for both particles), and each particle with individual *E/c* = 5 MeV/*c* and *mc* = 3 MeV/*c*. So $(E/c)_{total} = 5 + 5 = 10\ \text{MeV}/c$ and the total momentum $p_{xtotal} = -4 + 4 = 0\ \text{MeV}/c$. Figure 11(a) shows the *mc*, p_x and *E/c* of the individual particles. Figure 11(b) shows the total momentum $p_{x\ total}$, the total *E/c* and the total *mc* for the system of two particles. Solving for $(mc)_{total}$, we find that $(mc)_{total}{}^2 = (E/c)_{total}{}^2 - (p_{x\ total})^2 - (p_{y\ total})^2 - (p_{z\ total})^2 = 10^2 - 0^2 - 0^2 - 0^2 = 100$. This gives $(mc)_{total} = \sqrt{100} = 10\ \text{MeV/c}$ and so $m_{total} = 10\ \text{MeV}/c^2$. For a system of two (or more) masses moving relative to each other, their total mass is *greater than* the sum of their individual masses. This result is well-known in special relativity.

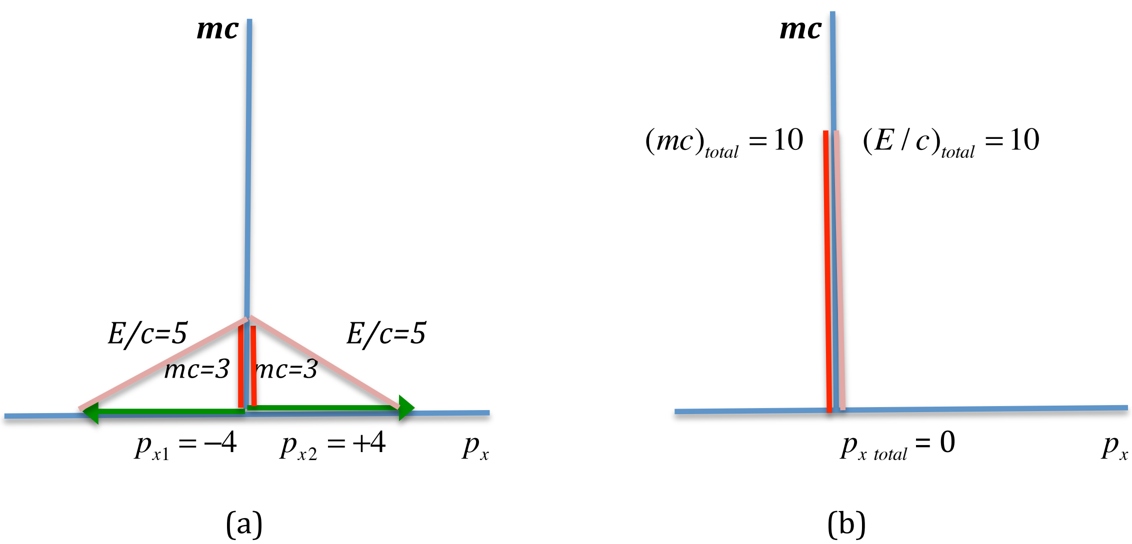

Figure 11. Modified Minkowski momenergy diagrams for two oppositely-moving particles. Figure 11(a) is for the two individual particles and Figure 11(b) is for the system of two particles. The total mass $m_{total} = 10\ \text{MeV}/c^2$ of the system is greater than the sum of the masses of the individual particles $m + m = 3 + 3 = 6\ \text{MeV}/c^2$.

What Are the Directions of Time and Energy in Spacetime?

Since the Modified Minkowski metric does not take time variable *ct* and energy variable *E/c* as fourth perpendicular dimensions for calculating the invariant quantities as does the Minkowski metric, can we still ask what are the directions of *ct* and *E/c* in spacetime? Consider the Modified Minkowski momenergy diagram of the energy-momentum equation $(E/c)^2 = p_x{}^2 + p_y{}^2 + p_z{}^2 + (mc)^2$. *E/c* is the hypotenuse of the triangle. Here $p_y = p_z = 0$ and $v_y = v_z = 0$, so $p_x = p$ and $v_x = v$. So we have $(E/c)^2 = p_x{}^2 + (mc)^2$ for our momenergy relation in the Modified Minkowski diagram (Figure 12).

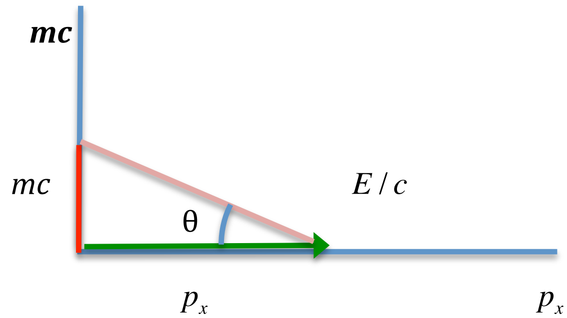

Figure 12. Modified Minkowski momenergy diagram where $p_y = p_z = 0$.

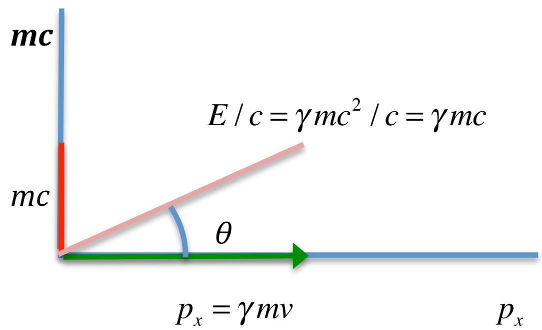

Figure 13. Modified Minkowski momenergy diagram of Figure 12 with hypotenuse *E/c* rotated. The angle θ is unchanged from Figure 12.

In Figure 13 the *E/c* hypotenuse has been rotated without changing its length *E/c*, and has been expressed in terms of the object's total energy $E/c = \gamma mc^2/c = \gamma mc$, while the relativistic momentum component p_x is expressed as $p_x = \gamma m v_x = \gamma m v$ (since $v_x = v$ here. The angle θ between E/c and p_x is indicated and is unchanged from Figure 12.

Now if we divide all three momentum vectors in Figure 13 by the quantity γmc, the diagram becomes Figure 14, showing the mathematical proportions of the triangle sides.

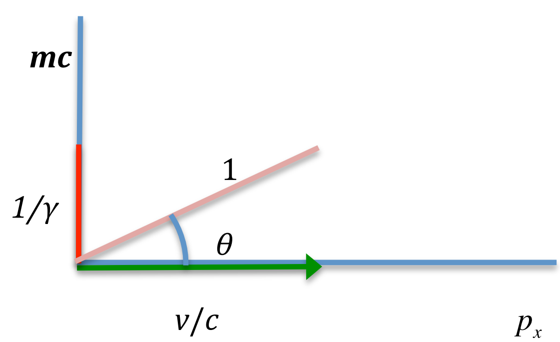

Figure 14. Proportions of sides of Modifed Minkowski momenergy triangle.

We can see from the Modified Minkowski proportion diagram in Figure 14 that

a) $\cos\theta = (v/c)/1 = v/c$
b) $\sin\theta = (1/\gamma)/1 = 1/\gamma$
c) $\tan\theta = (1/\gamma)/(v/c) = c/v\gamma$

Similarly if we divide all three lengths of the Modified Minkowski spacetime figure in Figure 15 below (corresponding to Figure 3 above) by the quantity $c\Delta t$, the diagram becomes Figure 16, showing the mathematical proportion of the sides.

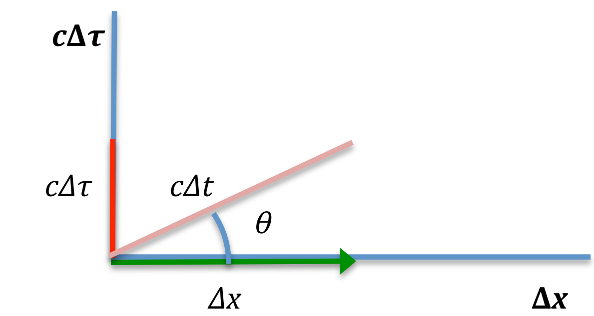

Figure 15. Modified Minkowski spacetime diagram, similar to Figure 3.

When we divide all three lengths of the Modified Minkowski spacetime diagram by $c\Delta t$, we get Figure 16.

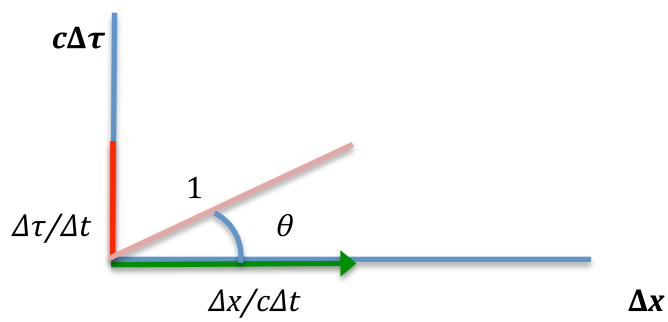

Figure 16. Proportions of the three sides of the Modified Minkowski spacetime triangle when all coordinates are divided by $c\Delta t$.

Since $\Delta x/\Delta t = v$ where v is the speed necessary to travel the distance Δx in time Δt, we have $\Delta x/c\Delta t = v/c$ in the figure. Also by the Pythagorean theorem, $\Delta\tau / \Delta t = \sqrt{1^2 - (v/c)^2} = \sqrt{1 - v^2/c^2} = 1/\gamma$. So Figure 16 above becomes Figure 17 below.

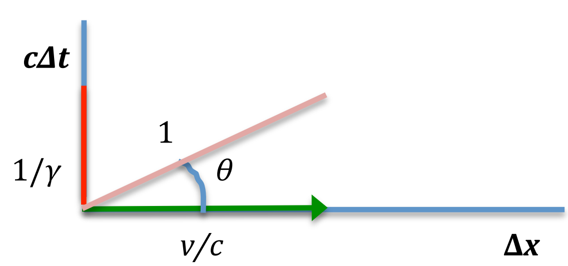

Figure 17. Proportions of the three sides of the Modified Minkowski Spacetime triangle.

We can see from the Modified Minkowski spacetime diagram in Figure 17 (as with Figure 14 for the Modified Minkowski momenergy diagram) that

a) cos $\theta = (v/c)/1 = v/c$
b) sin $\theta = (1/\gamma)/1 = 1/\gamma$
d) tan $\theta = (1/\gamma)/(v/c) = c/v\gamma$

Notice that the proportions of the sides of the Modified Minkowski spacetime triangle (Figure 17) and the proportions of the sides of the Modified Minkowski momenergy triangle (Figure 14) are the same. So if the quantity $c\Delta t$ (the hypotenuse in Figure 15) is known in a problem, the invariant spacetime interval $c\Delta\tau$ is found from the proportions in Figure 17 to be $c\Delta\tau = c\Delta t \sin\theta = c\Delta t \times 1/\gamma = c\Delta t/\gamma$.

Using the Modified Minkowski Spacetime Diagram with the "Twin Paradox".

Special relativity predicts that a twin that goes on a space voyage at near light speed v to another star and returns to Earth will not have aged as much as the twin that remained on Earth the whole time. (The full explanation of this effect takes into account the accelerations experienced by the travelling twin than are not experienced by the twin that stayed at home.)

Problem: Twin Stella takes a rocket ship travelling at v=0.990c to a star 4.00 light-years away from Earth, then turns around and returns to Earth. Twin Sally stays behind. How many years will Sally and Stella have aged when Stella returns?

Solution: A one-way trip takes Δt_1 = one-way distance/velocity = 4.00 light-years/0.990c = 4.04 years as measured by Sally on Earth. The round trip takes twice that: $\Delta t_{total} = 2\Delta t_1 = 2 \times 4.04 = 8.08$ years as measured by Sally. This is how much Sally will have aged during Stella's round trip to the star.

Figure 18 shows the Modified Minkowski spacetime diagrams for Stella's round trip to the star. In Figure 18(a) the hypotenuse $c\Delta t_1$ for the one-way trip is $c\Delta t_1 = c \times 4.04$ years $= 4.04$ light-years. For v = 0.990c the value of γ is $1/\sqrt{1-v^2/c^2} = 1/\sqrt{1-.990^2} = 7.089$ and the value of θ is $\cos^{-1}(v/c) = \cos^{-1}(.990)$ = 8.11 degrees. Using the angle θ (where cosθ = v/c) and the Modified Minkowski spacetime geometry of Figure 18(a), we see from the triangle in the diagram (where Δx is the distance from Earth to the star) that Stella's increase in age $\Delta\tau$ during her round trip (as measured by Sally) is obtained from the geometric relation for the triangle in Figure 18(a): $c\Delta\tau = 2\Delta x \tan\theta$ = 2 (4 c years)(1/γ)/(v/c)=(8/7.089)/.990 = 1.14 c years . So the increase in Stella's age on returning to Earth is $\Delta\tau$ = 1.14 c years/c = 1.14 years. Figure 18(b) shows that Sally has aged 8.08 years, since for her, $(c\Delta t)^2 = (c\Delta\tau)^2 + (\Delta x)^2 = (c\Delta\tau)^2 + (0)^2 = (c\Delta\tau)^2$. Therefore $\Delta\tau = \Delta t$ for Sally because Sally's own displacement Δx during Stella's round trip to the star was zero.

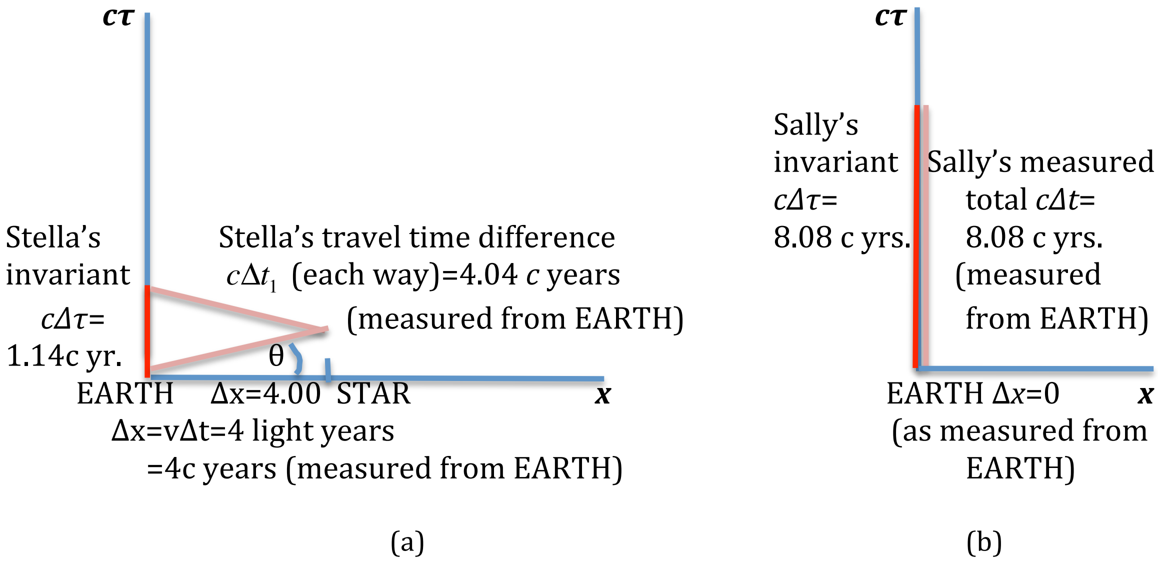

Figure 18. Solving the "twin paradox" problem using the Modified Minkowski spacetime diagram. (a) Stella makes a round trip to a star 4 light-years away while travelling each way at v = 0.990c, with cosθ= v/c = 0.990 , so θ=8.11 degrees. Stella's one-way distance is 4 light years and her one-way travel time is 4.04 years as measured from Earth. Stella has aged the amount of her invariant time of $\Delta\tau$=1.14 years during her round trip. (b) Sally's age has increased by 8.08 years during Stella's round trip. Sally's invariant time equals her measured time for Stella's round trip because Sally's Δx=0 because she remained on Earth during Stella's round trip to the star.

Are Time and Energy in the Fourth Dimension?

The special theory of relativity shows that time and space are closely interrelated although they are not interchangeable. The idea that time and space will have less meaning separately and more meaning when considered as aspects of a unified spacetime owes much to Minkowski and Minkowski diagrams of spacetime relationships. Einstein relied on Minkowski's mathematical approach to spacetime when developing his own general theory of relativity. This theory, which has much experimental and astronomical support, relates gravity to a warping of spacetime produced by concentrations of energy and momentum, and incorporates the Minkowski metric as well as more complex metrics.

In the Minkowski metric, E/c (having the units of of momentum) is orthogonal to the other three components p_x, p_y and p_z (or the three-vector **p**). These four orthogonal momentum components combine by a kind of negative Pythagorean theorem to produce the invariant quantity *mc*.

In the Modified Minkowski momenergy metric, the 4-vector component *mc* is orthogonal to p_x, p_y and p_z (or ***p***), while the quantity E/c now has the magnitude of the vector sum of the orthogonal components *mc*, p_x, p_y and p_z by means of the actual Pythagorean theorem (applied in four dimensions). In the Modified Minkowski metric, the hypotenuse E/c makes an angle θ with the vector ***p*** given by cosθ = v/c, so θ can be anywhere from 0 degrees to 90 degrees. In the Modified Minkowski metric the angle θ between E/c and ***p*** is only 90 degrees when ***p***=0.

Similarly, in the Minkowski spacetime metric, the *cΔt* component is orthogonal to the *Δx*, *Δy* and *Δz* components, producing the invariant interval *Δs*=*cΔτ* by the negative Pythagorean theorem. But in the Modified Minkowski spacetime metric, *Δs*=*cΔτ* is one of the orthogonal components of the difference vector $X_2 - X_1$ (along with *Δx*, *Δy* and *Δz*) while *cΔt* is the actual Pythagorean hypotenuse (applied in four dimensions). As seen in Figure 15 above, *cΔt* makes an angle *θ* with Δx (when *Δy*=*Δz*=0) where cos*θ* = *v/c*. So the angle *θ* between *cΔt* and *Δx* (or the 3-d displacement vector ***r***) in the Modified Minkowski metric can be anywhere between 0 degrees and 90 degrees, and is only 90 degrees when *Δx* (or ***r***) is zero.

The Modified Minkowski metric contains the invariant spacetime interval *cΔt* and the invariant momentum *mc* as 4th dimensional quantities, displacing the variable quantities *ct* and *E/c* from their 4th dimensional status in the Minkowski metric. Whether this modification of the Minkowski metric will lead to a reconsideration of the nature of time (and energy) as 4th dimensional quantities remains to be seen.

The Origin of the Modified Minkowski Metric

The reader may wonder about the origin of the Modified Minkowski metric. I am proposing this metric based on the momentum relationships in the Gauthier[5] model of a relativistic electron that is composed of a circulating spin-1/2 charged photon.

Briefly, a circling charged photon-like object proposed to compose an electron of rest energy $E_o = mc^2$ has circling momentum $P = E_o / c = mc$. If this electron model moves relativistically with velocity *v* in a direction perpendicular to the plane of the charged photon's circular motion, the charged photon's energy (since it is the electron's relativistic energy) becomes $E = \gamma mc^2$ and its momentum becomes $P = E / c = \gamma mc$. The circling charged photon composing a resting electron becomes a helically-moving charged photon that has longitudinal momentum components p_x, p_y and p_z and a transverse momentum component $p_{transverse} = mc$. The forward angle *θ* of the charged photon's helical trajectory is given by cos *θ* = *v/c.* The transverse and longitudinal momentum components of the helically moving charged photon-like lead to the Pythagorean equation $p_{transverse}{}^2 + p_{longitudinal}{}^2 = p_{total}{}^2$, or $(mc)^2 + p_x{}^2 + p_y{}^2 + p_z{}^2 = (E / c)^2$, or $(mc)^2 + p^2 = (E / c)^2$. This result is both the relativistic energy-momentum equation, and the Pythagorean sum of squares of the components of a 4-vector (mc, p_x, p_y, p_z) equaling $(E / c)^2$. This was the origin of the Modified Minkowski 4-momentum metric relationship. The above momentum relationships are illustrated on the Modified Minkowski momenergy diagram in Figure 19 below.

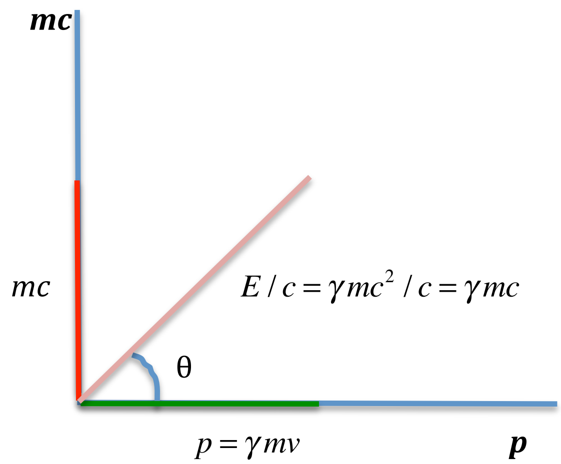

Figure 19. Modified Minkowski momenergy 4-vector (mc, p_x, p_y, p_z) whose components add according to the Pythagorean theorem to give the total diagonal momentum vector *E/c*. θ (where $\cos\theta = v/c$) is the angle between the *E/c* momentum vector and the *p* momentum vector.

When Figure 19 is compared with Figure 20 below, which is from Gauthier[5], the reader can immediately see why the Modified Minkowski 4-momentum metric (mc, p_x, p_y, p_z) suggested itself to the author. The Modified Minkowski spacetime metric $(c\tau, x, y, z)$ was developed soon after the Modified Minkowski momenergy metric.

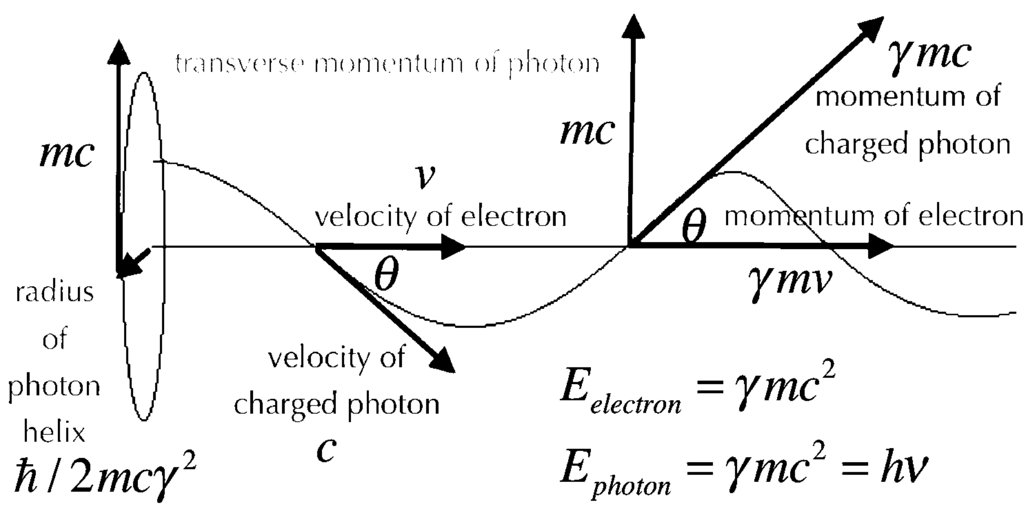

Figure 20. (From Gauthier[5]) Velocity, momentum and energy relationships for a relativistic electron modeled by a proposed helically-circulating spin-1/2 charged photon. The velocity and momentum vectors and vector components of the charged photon are indicated.

The proposed model of the electron as a circulating spin-1/2 charged photon gives a physical meaning to the momentum quantities *mc* and $E/c = \gamma mc$ in Minkowski and Modified Minkowski momenergy diagrams for an electron. As seen in Figure 20 above, *mc* is the momentum of the circling spin-1/2 charged photon proposed to compose a resting electron. $E/c = \gamma mc$ is the total momentum of the helically-circulating spin-1/2 charged photon composing the moving electron model, having *mc* as its transverse component of momentum and $p = \gamma mv$ as its longitudinal component of momentum. Similarly, the distances $c\Delta\tau$ and $c\Delta t$ in the Minkowski and Modified Minkowski spacetime diagrams have physical meanings in the proposed electron model. $c\Delta\tau$ is the distance traveled during the proper time $\Delta\tau$ by the circling charged spin-1/2 charged photon. The distance $c\Delta t$ is the distance the helically moving spin-1/2 charged photon travels along its helical trajectory during the time Δt measured between two events associated with the moving electron. The Modified Minkowski metric is however independent of any particular microscopic model of matter. The Modified Minkowski metric, like the Minkowski metric, stands on the correctness of the experimentally well-established relativistic invariance relations for spacetime and mass-energy.

Further Discussion

The proposed Modified Minkowski metric for spacetime and for momenergy is a new way to geometrically represent the mathematical relations among space, time, energy and momentum in Einstein's special theory of relativity. It proposes a small but significant modification in the Minkowski 4-vectors for spacetime and for momenergy. The Minkowski metric has been used since 1908 to diagram spacetime and momenergy relations to help illustrate invariance relations in special relativity and to solve special relativity problems. Although the Minkowski metric has been very successful and is still highly utilized, this geometrical method is somewhat awkward. In reference to Minkowski's mathematical approach to relativity, Einstein[6] is quoted as saying, "Since the mathematicians have invaded the theory of relativity I do not understand it myself any more." The Minkowski metric requires hyperbolic mathematics, rather than basic trigonometry and Pythagorean geometry used with the Modified Minkowski metric, to graphically display relativistic relations and the mathematical invariant spacetime and momenergy relations of special relativity.

The Minkowski spacetime and momenergy metrics set the time *t* (multiplied by *c*) and the total energy *E* (divided by *c*) respectively as the 4th dimension components in the two metrics, along with the three dimensions of space (*x,y,z*) and the corresponding three spatial dimensions of momentum (p_x, p_y, p_z). The Modified Minkowski metric replaces *ct* by the invariant spacetime interval $c\tau$ in the 4-dimensional spacetime metric and replaces *E/c* by the invariant momentum *mc* in the 4-dimensional momenergy metric. The result is that the squares of the time term *ct* and the energy term *E/c* in the Modified Minkowski metrics become equal to the simple Pythagorean sum of squares of the 4 components of the Modified Minkowski metrics for spacetime and momenergy respectively. The mathematical invariance relations of special relativity are not altered by this modification of the Minkowski metric. But the spacetime and momenergy diagrams are simplified.

Minkowski's spacetime metric has found great utility in Einstein's general theory of relativity, where gravity is interpreted as curved spacetime. There may be implications of the Modified Minkowski metric for general relativity, since the Minkowski metric applies in general relativity in regions of curved spacetime that are sufficiently localized so that spacetime is locally flat. In curved spacetime the invariant special relativity differential invariant interval *ds* is found from the formula $(ds)^2 = (cdt)^2 - dx^2 - dy^2 - dz^2$. This mathematical relationship is used both by Minkowski diagrams and by Modified Minkowski diagrams. It will be interesting to see in general relativity if the Modified Minkowski spacetime metric can add greater simplicity than the Minkowski spacetime metric, in the way that the Modified Minkowski metric simplifies special relativity's spacetime and momenergy diagrams.

Are there any logical reasons why the traditional Minkowski metric, which takes time t (or more precisely the distance ct) and total energy E (or more precisely the momentum E/c) as 4^{th} dimensional quantities, should be preferred to the proposed Modified Minkowski metric, which sets the invariant spacetime interval $c\Delta\tau$ and the corresponding invariant momentum mc as 4^{th} dimensional quantities? Both metrics conform to the equations of special relativity and are useful in solving problems in special relativity. Minkowski spacetime and momenergy diagrams are more complicated than Modified Minkowski diagrams because Minkowski diagrams geometrically represent special relativity's invariant spacetime and energy-momentum relations by the use of hyperbolic or negative Pythagorean relationships. Modified Minkowski spacetime and momenergy diagrams have a greater simplicity gained by geometrically representing special relativity's invariant spacetime and energy-momentum relationships by using the actual Pythagorean theorem. This article has introduced the Modified Minkowski metric with several illustrative comparisons of Minkowski and Modified Minkowski diagrams. Readers familiar with using traditional Minkowski diagrams for solving special relativity problems could generate other examples comparing the two approaches.

Conclusions

The proposed Modified Minkowski metric provides an alternative way to make geometrically-based spacetime and momenergy diagrams to better understand special relativity and to help solve special relativity problems, without modifying the invariance relations found experimentally and described mathematically by special relativity. Any techniques that facilitate the teaching of relativity without compromising the equations of relativity should be welcomed by physics teachers. If there are further benefits from using the Modified Minkowski metric such as a possibly deeper understanding of the nature of space and time, these benefits should be welcomed as well.

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