APPENDIX B - DERIVATION OF THE RELATIVISTIC ENERGY-MOMENTUM EQUATION

Reference [28] mentions on page 835 that simply combining equations (5.62) and (5.65) to each other should generate the complete relativistic energy-momentum equation (5.70), but they do not offer the detailed derivation of this equation: $E^2 = (pc)^2 + (mc^2)^2$.

Here is for convenience the complete step by step derivation of this famous equation:

$$E = \gamma m_0 c^2 \qquad (5.62)$$

$$\frac{E}{m_0 c^2} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\left(\frac{E}{m_0 c^2}\right)^2 = \frac{1}{1 - v^2/c^2}$$

$$\frac{E^2}{m_0{}^2 c^4} = \frac{1}{1 - v^2/c^2} \qquad (B1)$$

$$p = \gamma m_0 v \qquad (5.65)$$

$$\frac{p}{m_0 v} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\left(\frac{p}{m_0 v}\right)^2 \frac{v^2}{c^2} = \frac{v^2/c^2}{1 - v^2/c^2}$$

$$\frac{p^2}{m_0{}^2 c^2} = \frac{v^2/c^2}{1 - v^2/c^2} \qquad (B2)$$

Subtracting momentum equation (B2) term for term from mass equation (B1), we obtain:

$$\frac{E^2}{m_0{}^2 c^4} - \frac{p^2}{m_0{}^2 c^2}\frac{c^2}{c^2} = \frac{1}{1 - v^2/c^2} - \frac{v^2/c^2}{1 - v^2/c^2} \qquad (B3)$$

$$\frac{E^2}{m_0{}^2 c^4} - \frac{p^2 c^2}{m_0{}^2 c^4} = \frac{1}{1 - v^2/c^2} - \frac{v^2/c^2}{1 - v^2/c^2}$$

$$\frac{E^2 - p^2 c^2}{m_0{}^2 c^4} = \frac{1 - v^2/c^2}{1 - v^2/c^2} = \frac{\gamma}{\gamma} \qquad (B4)$$

$$\frac{E^2 - p^2 c^2}{m_0{}^2 c^4} = \frac{\gamma^2}{\gamma^2}$$

$$\gamma^2 \left(E^2 - p^2 c^2\right) = \gamma^2 m_0{}^2 c^4$$

$$\gamma^2 E^2 - \gamma^2 p^2 c^2 = \left(mc^2\right)^2 \text{ where } m = \gamma m_o$$

$$\gamma^2 E^2 = p^2 c^2 + \left(mc^2\right)^2 \text{ where } p = \gamma m_o v \tag{B5}$$

And finally, since γ is dimensionless, then E=γE and we obtain equation (5.70):

$$E^2 = \left(pc\right)^2 + \left(mc^2\right)^2 \tag{5.70}$$

With equation (B4), there is a strong temptation to simplify both occurrences of the Lorentz factor to 1 before proceeding, but this leads to the often encountered non-relativistic version: $E^2 = (pc)^2 + (m_0 c^2)^2$ which is Newtonian and adds only the translational half of the kinetic energy of a massive particle in motion, leaving out the electromagnetically oscillating half that converts to the relativistic mass increment (see Section 3.7).

The proper procedure is to square the mutually reducible γ factor occurrences, so they can be reunited with the two occurrences of m_o as the development proceeds.

At face value, fusing the last occurrence of the squared γ factor with the energy ($\gamma^2 E^2$) may seem to be problematic, but considering that this factor is a dimensionless quantity, it can be multiplied with the energy component without any adverse effect for the integrity of the equation.

(Abstracted from Electromagnetic Mechanics of Elementary Particles)